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Derived formal moduli problems

following Jacob Lurie

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To my high school teacher,
Pietro Cerruti

Introduction

This mémoire has two main goals: on one side, we try to give a significant account of the theory of $(\infty, 1)$-categories using the language of quasicategories of Joyal; on the other side, we describe an application of the techniques developed. Let us present the first part (which is exposed in Chapter 2).

Starting from the foundational work of D. Quillen [Qui67], the language of homotopical algebra reached several areas of Mathematics; among the main applications we can recall the earliest successes with the construction of the cotangent complex and the higher K-theory. More recently, it has been understood that in a geometric context the model categories of Quillen are simply needed as a presentation of a subtler object, which encodes the same amount of information of the original category, but in a more essential way. It is a fact that the theory of model categories is redundant: fibrations and cofibrations are nothing but a technical tool making certain theorems to be true. For example, the construction of the homotopy category does not depend in any way on fibrations and cofibrations; however, to prove that the homotopy category exists within the same universe we started with, one needs to use them. In practice, we need those two additional classes of maps in order to coherently define cylinder and path objects, which are used to deal with (higher) homotopies. However, there is another way to construct such objects: it is a standard fact that inside a simplicial model categories the enrichment can be used in order to produce cylinder and path objects of any order, at least after replacing the category with a suitable subcategory. Following this reasoning, one might be led to expect that the simplicial enrichment is enough to reconstruct the model structure or, at least, to reproduce the same amount of information of the original model structure, in a sense to be made precise. In fact, there is some evidence: it is a well-known result that weak equivalences can be recognized using mapping spaces in a general model category. Actually, there is more: the existence of mapping spaces[1] naturally leads one to think that "model categories are simplicially enriched up-to-homotopy". This statement can be made more precise and can be proven using the technique called Dwyer-Kan localization (we will deal with it in Chapter 2). The important lesson we can learn from this is that there is a close relation between model categories and simplicially enriched categories; this relation can be used in order to get a grasp on the theory of $(\infty, 1)$-categories.

Roughly speaking an $(\infty, 1)$-category is a category with morphisms "of every order" satisfying the additional condition that every n-morphism is invertible for $n > 1$. The correct formalization of this idea took several years to the mathematical community; one of the easiest ways of formulating such a theory is to consider categories enriched over **CHaus** (the category of compactly generated Hausdorff spaces), in such a way that 1-morphisms are paths, 2-morphisms are homotopies (with fixed ending points) and so on. This approach is not really useful, as it is quite hard to deal with. A better choice is to consider simplicially enriched categories: it is in fact known from a long time that simplicial sets can be used to describe **CHaus** in any question only concerned with homotopies. This suggests that the theory of topological categories is equivalent to the theory of simplicially enriched category (and it is not harder to prove formally this statement, up to giving a rigorous definition of what equivalent means in this context). Therefore, assuming the model of simplicial categories as model for the theory of $(\infty, 1)$-categories, we see that the Dwyer-Kan localization gives a way to associate to every model category an $(\infty, 1)$-category. The quasicategories of Joyal represent another approach to the same theory. In this mémoire we will describe $(\infty, 1)$-categories from this point of view, developing (or at least sketching)

[1] We are referring for the moment to the construction using simplicial and cosimplicial resolutions.

generalization of standard categorical constructions to this new setting.

The second part of this mémoire deals with an application of this language to Derived Algebraic Geometry (which can be thought both in the sense of Toën and Vezzosi and in the sense of Lurie). More specifically, we will be concerned with a derived version of (formal) deformation theory: in classical deformation theory one is concerned with the properties of deformation functors (which are a particular class of functors of Artin rings). It is possible to define a similar class of objects in the derived setting, leading to the notion of (derived) formal moduli problems, which can be thought essentially as formal neighbourhoods of a point in a derived stack. It is possible to attach to every such formal moduli problem a "tangent complex", which carries in a natural way a structure of differential graded Lie algebra. This can be intuitively explained as follows: the formation of the tangent complex is an operation which commutes with (homotopy) limits; it follows that $T_{\Omega X} \simeq \Omega T_X$, where Ω denotes the suspension functor (which can be defined in every pointed $(\infty, 1)$-category) and the object ΩX should be thought as an internal group object (up to homotopy), so that $T_{\Omega X}$ should carry a Lie algebra structure. The main result of this mémoire is to show that in characteristic 0, such a Lie algebra structure is enough to reconstruct completely the formal moduli problem we started with. A more precise formulation is given in Theorem 4.4.1.

Acknowledgements

The story of this mémoire began at least one year ago when Professor Luca Barbieri Viale told me that Gabriele Vezzosi was about to move from Florence to Paris. At that time, I was becoming familiar with the Algebraic Geometry in the sense of Grothendieck, and I was contemplating those wonderful ideas with an astonished look. I had already heard of what Gabriele Vezzosi was doing, together with Bertrand Toën, and their papers were sparkling in front of my eyes, even though I wasn't really able to understand them. Since I was moving to Paris, I decided to ask Gabriele Vezzosi to accept me as master student. He accepted, and so I began to work on this mémoire around November 2012.

It took me all this time to master the basic techniques of this discipline, which is called *Homotopical Algebraic Geometry*, nor I would have been able to do so without the help and the guidance of Gabriele Vezzosi. I am especially grateful to him for giving his students the opportunity to teach several seminars concerning the basics of HAG; I learned a lot of amazing ideas and techniques in preparing the notes and the talks he assigned to me. And, besides that, this inserted me in a really beautiful context; I have been undoubtedly enriched from the collaboration with the other students. This is why I am thanking Gabriele above anyone else, but also the participants to the groupe de travail *Autour de la Géométrie Algébrique Dérivée.*

Another debt I have is toward Professor Luca Barbieri Viale. He guided me during last year, when I was moving my first steps in the realm of Algebraic Geometry, giving me several wise advices. Also, he has been the first one who taught me about model categories, which have been proved extremely useful during this year. Therefore, it is a pleasure for me to express my deep gratitude to him.

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Orsay,
June 2013,
Mauro Porta

Contents

Chapter 1

Classical formal deformation theory

The goal of this first chapter is to introduce the reader to the context of classical deformation theory. We tried to avoid the abstract language that will characterize the following chapters. However, the need for the language of extensions forced us to introduce the cotangent complex from the very beginning, in order to make the development of the subject more natural and elegant. In order to make it compatible with our presentation, we construct it from the scratch using, *however*, an *ad hoc* definition. We will return on this subject later, reviewing it inside a more general framework which makes use of the techniques of Quillen.

1.1 Cotangent complex

The reader is assumed to be familiar with the language of standard category theory and with the language of homological algebra. More in details, the prerequisites for this section are:

1. the theory of monads (the reference is [Mac71, Ch. VI]);
2. the standard simplicial resolution associated to a monad (see [Wei94, Ch. 8.6]);
3. the Dold-Kan correspondence (see [Wei94, Ch. 8.4]);
4. the notion of derived category of an abelian category and more generally the basic techniques of triangulated categories (see [Wei94, Ch. 10]).

1.1.1 The setting

The cotangent complex is an object associated to a morphism of schemes $f\colon X \to Y$ living in the derived category of quasi–coherent sheaves over X. It is related to the notion of smoothness and from a certain point of view it generalizes the construction of Kähler differentials. When the morphism f is smooth, the cotangent complex gives back exactly the same amount of information of the sheaf of Kähler differentials; however, when f is not smooth, the cotangent complex is a better tool to analyze the behavior of f. This is the same phenomenon that happens when dealing with Grothendieck duality: when the maps we are considering are smooth enough (i.e. Cohen-Macaulay), the relative dualizing sheaf exists; however, when the smoothness isn't enough, the best we can do is to construct an object in the derived category of quasi-coherent sheaves.

In this section we will use the (pointwise) Dold-Kan correspondence in order to switch from bounded complexes of quasi-coherent sheaves to simplicial objects in $\mathbf{QCoh}(X)$.

Definition 1.1.1. Let $(X, \mathscr{O}_X)$ be a scheme. A simplicial commutative $\mathscr{O}_X$-module is a simplicial object in $\mathscr{O}_X$-**Mod**. The category of simplicial $\mathscr{O}_X$-modules will be denoted $\mathscr{O}_X$-**sMod**. We will say that a simplicial $\mathscr{O}_X$-module is quasi-coherent if it is quasi-coherent levelwise. We will denote by $\mathbf{sQCoh}(X)$ the category of quasi-coherent simplicial $\mathscr{O}_X$-modules.

Remark 1.1.2. It is clear that a quasi-coherent simplicial $\mathscr{O}_X$-module is the same as a simplicial object in $\mathbf{QCoh}(X)$.

Remark 1.1.3. Recall that the Dold-Kan correspondence produces, *for any abelian category* $\mathcal{A}$ an equivalence between $\mathrm{s}\mathcal{A}$ (the category of simplicial objects in $\mathcal{A}$) and $\mathbf{Ch}_{\leq 0}(\mathcal{A})$. In particular, we obtain an equivalence between $\mathscr{O}_X$-**sMod** and $\mathbf{Ch}_{\leq 0}(\mathscr{O}_X\text{-}\mathbf{Mod})$. Similarly, we have an equivalence between $\mathbf{sQCoh}(X)$ and $\mathbf{Ch}_{\leq 0}(\mathscr{O}_X\text{-}\mathbf{Mod})$.

Let $(X, \mathscr{O}_X)$ be a scheme. We have a functor

$$\mathcal{G} \colon \mathbf{sQCoh}(X) \to \mathrm{sSh}(X)$$

where $\mathrm{sSh}(X)$ is the category of simplicial sheaves on X, which forgets the structure of $\mathscr{O}_X$-module. This functor has a left adjoint:

Notation 1.1.4. If S is a set and R is a ring, we denote by $R^{(S)}$ the free R-module with basis S. In other words, $R^{(S)}$ is the image of S under the left adjoint to the forgetful functor R-**Mod** $\to$ **Set**.

Lemma 1.1.5. *Define* $\mathcal{F} \colon \mathrm{sSh}(X) \to \mathbf{sQCoh}(X)$ *by sending a simplicial sheaf* $\mathscr{F}_*$ *to the (levelwise) sheafification of the simplicial presheaf*

$$U \mapsto \mathscr{O}_X(U)^{(\mathscr{F}_*(U))}$$

Then $\mathcal{F}$ *is left adjoint to* $\mathcal{G}$*.*

Proof. We obviously have a natural transformation $\eta \colon \mathrm{Id}_{\mathrm{sSh}(X)} \to \mathcal{G}\mathcal{F}$ which "inserts the generators". Moreover, the claimed adjunction holds at level of presheaves. The universal property of sheafification implies that the same holds for sheaves. ☐

We pass now to simplicial commutative $\mathscr{O}_X$-algebras. Recall the following standard result:

Proposition 1.1.6. *Let* $(X, \mathscr{O}_X)$ *be a scheme. The category* $\mathbf{QCoh}(X)$ *of quasi-coherent sheaves on* X *has a symmetric monoidal structure* $(\mathbf{QCoh}(X), \otimes_{\mathscr{O}_X}, \mathscr{O}_X)$*, where* $\otimes_{\mathscr{O}_X}$ *is the tensor product of* $\mathscr{O}_X$*-modules.*

Proof. It is nothing more than a straightforward verification. ☐

Definition 1.1.7. Let $(X, \mathscr{O}_X)$ be a scheme. A (commutative) $\mathscr{O}_X$-algebra is a (commutative) monoid object in $(\mathbf{QCoh}(X), \otimes_{\mathscr{O}_X}, \mathscr{O}_X)$. The category of commutative $\mathscr{O}_X$-algebras will be denoted $\mathscr{O}_X$-**Alg**.

Definition 1.1.8. Let $(X, \mathscr{O}_X)$ be a scheme. A simplicial commutative $\mathscr{O}_X$-algebra is a simplicial object in the category $\mathscr{O}_X$-**Alg**. We will denote the category of simplicial commutative $\mathscr{O}_X$-algebras by $\mathscr{O}_X$-**sAlg**.

Let $(X, \mathscr{O}_X)$ be a scheme. We have another obvious forgetful functor

$$\mathcal{V} \colon \mathscr{O}_X\text{-}\mathbf{sAlg} \to \mathbf{sQCoh}(X)$$

This functor has a left adjoint:

Lemma 1.1.9. *Define* $\mathcal{S} \colon \mathbf{sQCoh}(X) \to \mathscr{O}_X\text{-}\mathbf{sAlg}$ *to be the symmetric algebra functor applied levelwise. Then* $\mathcal{S}$ *is a left adjoint for* $\mathcal{V}$*.*

It is well known that the analogous statement for the forgetful functor $\mathbf{QCoh}(X) \to \mathscr{O}_X$-**Alg** is true. Therefore previous lemma readily descends from the following categorical argument:

Proposition 1.1.10. *Let* $F \colon \mathcal{C} \rightleftarrows \mathcal{D} \colon G$ *be an adjunction between categories and let* $\mathcal{B}$ *be any other category. Then we have an adjunction*

$$F_* \colon \mathbf{Funct}(\mathcal{B}, \mathcal{C}) \rightleftarrows \mathbf{Funct}(\mathcal{B}, \mathcal{D}) \colon G_*$$

Proof. Let, in general, $\varphi\colon f \to g$ be a natural transformation between functors from $\mathcal{C}$ to $\mathcal{D}$. Then we can define a new natural transformation $\varphi_*\colon f_* \to g_*$ in the obvious way: if $h\colon \mathcal{B} \to \mathcal{C}$ is any functor, define

$$(\varphi_*)_h := \varphi_h\colon f \circ h \to g \circ h$$

It's clear that the construction of φ_* preserves both vertical composition and identities, so that $\mathbf{Funct}(\mathcal{B}, -)$ can be really regarded as a functor. This implies trivially that the adjunction $F \dashv G$ lifts to $F_* \dashv G_*$, since the triangular identities are preserved by $\mathbf{Funct}(\mathcal{B}, -)$. $\square$

Notation 1.1.11. Let $(X, \mathscr{O}_X)$ be a scheme. Consider the functors $\mathcal{F}$ and $\mathcal{S}$ introduced in Lemma 1.1.5 and 1.1.9 respectively. Then the composition $\mathscr{O}_X[-] := \mathcal{S} \circ \mathcal{F}$ gives a functor

$$\mathscr{O}_X[-]\colon \mathrm{sSh}(X) \to \mathscr{O}_X\text{-}\mathbf{sAlg}$$

which is a left adjoint for the obvious forgetful functor $\mathcal{U}\colon \mathscr{O}_X\text{-}\mathbf{sAlg} \to \mathrm{sSh}(X)$.

1.1.2 Canonical resolution

Let $(X, \mathscr{O}_X)$ be a scheme and consider the adjuntion

$$\mathscr{O}_X[-]\colon \mathrm{sSh}(X) \rightleftarrows \mathscr{O}_X\text{-}\mathbf{sAlg}\colon \mathcal{U}$$

introduced in Notation 1.1.11; let moreover η and ε be the unit and the counit of this adjunction. We canonically obtain a comonad $(\perp, \varepsilon, \delta)$ on $\mathscr{O}_X\text{-}\mathbf{sAlg}$, where

$$\perp := \mathscr{O}_X[\mathcal{U}(-)], \quad \delta := \mathscr{O}_X[\eta_{\mathcal{U}}]$$

The general techniques explained in [Wei94, Ch. 8.6] allows to construct a functor

$$\perp_*\colon \mathscr{O}_X\text{-}\mathbf{Alg} \to \mathscr{O}_X\text{-}\mathbf{sAlg}$$

associating to an $\mathscr{O}_X$-algebra $\mathscr{A}$ the simplicial set

$$\{\perp_n \mathscr{A}\}_{n \in \mathbb{N}}$$

where

$$\perp_n \mathscr{A} := \perp^{n+1} \mathscr{A}$$

and where face and degeneracy operators are defined as

$$\begin{aligned} \partial_i &:= \perp^i \varepsilon \perp^{n-i}\colon \perp^{n+1}\mathscr{A} \to \perp^n \mathscr{A} \\ \sigma_i &\colon \perp^i \delta \perp^{n-i}\colon \perp^{n+1}\mathscr{A} \to \perp^{n+2}\mathscr{A} \end{aligned}$$

The natural morphism $\varepsilon_{\mathscr{A}}\colon \perp\mathscr{A} \to \mathscr{A}$ satisfies the identity

$$\varepsilon_{\mathscr{A}} \circ \partial_0 = \varepsilon_{\mathscr{A}} \circ \partial_1 \tag{1.1}$$

so that this simplicial object is augmented. Moreover, we have the following key result:

Proposition 1.1.12. *Let $\mathscr{A}$ be an $\mathscr{O}_X$-algebra. The augmented chain complex corresponding to $\perp_*\mathscr{A}$ via Dold–Kan is exact.*

Proof. See [Wei94, Proposition 8.6.8]. $\square$

1.1.3 The construction

At this point we can easily construct the cotangent complex associated to a morphism of schemes.

Definition 1.1.13. Let $f\colon X \to Y$ be a morphism of schemes. Let $\perp$ be the monad associated to $f^{-1}\mathscr{O}_Y\text{-}\mathbf{sAlg}$ as in 1.1.2. The cotangent complex of f is the simplicial $\mathscr{O}_X$-module

$$\mathbb{L}_{X/Y} := \Omega^1_{\perp_*\mathscr{O}_X / f^{-1}\mathscr{O}_Y} \otimes_{\perp_*\mathscr{O}_X} \mathscr{O}_X$$

where the construction of the sheaf of Kähler differentials is done levelwise.

Equation (1.1) translates in this context into an augmentation

$$\perp_* \mathscr{O}_X \to \mathscr{O}_X$$

which induces another augmentation morphism

$$\mathbb{L}_{X/Y} \to \Omega^1_{\mathscr{O}_X/f^{-1}\mathscr{O}_Y} \otimes_{\mathscr{O}_X} \mathscr{O}_X \simeq \Omega^1_{X/Y}$$

This induces an isomorphism

$$\mathscr{H}_0(\mathbb{L}_{X/Y}) \to \Omega^1_{X/Y}$$

Remark 1.1.14. Let $(\mathcal{C}, \tau)$ be a Grothendieck site and let $\mathscr{F} \in \mathrm{sSh}(\mathcal{C})$ be a simplicial sheaf. Following Jardine [Jar] we consider the sheaves of homotopy groups of $\mathscr{F}$. In our setting the definition given by Jardine coincides with the definition of homotopy group of a simplicial object in an abelian category given in [Wei94, Definition 8.3.6]. Then [Wei94, Theorem 8.3.8] can be used to deduce that

$$\pi_0(\mathbb{L}_{X/Y}) \simeq \Omega^1_{X/Y}$$

where π_0 is meant in the sense of Jardine.

The cotangent complex has a functorial behavior which is formally similar to the one of $\Omega^1_{X/Y}$. The main result in this direction is the existence of the transitivity triangle:

Theorem 1.1.15. *Let $f \colon X \to Y$ and $g \colon Y \to Z$ be maps of schemes. Then there is a canonical exact triangle in $D(X)$*

$$\begin{tikzcd} & \mathbb{L}_{X/Y} \arrow[dl, tail] & \\ f^*\mathbb{L}_{Y/Z} \arrow[rr] & & \mathbb{L}_{X/Z} \arrow[ul] \end{tikzcd}$$

Proof. To be added. □

Corollary 1.1.16. *Let $f \colon X \to Y$ and $g \colon Y \to Z$ be maps of schemes. Then there is a long exact sequence of sheaves on X*

$$f^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0$$

Proof. This is the last part of the long exact sequence associated to the exact triangle of Theorem 1.1.15. □

1.1.4 The language of extensions

We apply now the theory of cotangent complex to develop the language of extensions, which is of crucial importance in deformation theory.

Definition 1.1.17. Let A be a commutative ring. An *(A-)extension* of an A-algebra B is a surjective morphism of A-algebras $f \colon C \to B$ such that $I = \ker f$ is a nilpotent ideal of B'. If moreover $I^2 = 0$, then the extension is said to be a *square-zero extension.*

Definition 1.1.18. Let A be a commutative ring and let (B_1, I_1, f_1), (B_2, I_2, f_2) be two A-extensions of B. A morphism of A-extensions is simply a morphism of A-algebras $g \colon B_1 \to B_2$ such that $f_1 = f_2 g$.

Let A be a fixed commutative ring and consider an A-algebra B. We can obviously define a category $\mathbf{Ex}_A(B)$ whose objects are the A-extensions of B and whose morphisms are the morphism of extensions.

One has the following result:

Lemma 1.1.19. *Let A be a commutative ring and let $f \colon B' \to B$ be a square-zero extension of B by I. Then I has a structure of B-module.*

Proof. Let $m \in I$ and $b \in B$. We define $b \cdot m := b'm$, where $b' \in B'$ is any element satisfying $f(b') = b$. It is straightforward to check that this definition gives a B-module structure over I. □

The previous lemma allows us to construct a forgetful functor

$$\mathcal{U}\colon \mathbf{Ex}_A(B) \to B\text{-}\mathbf{Mod}$$

sending an extension (B', I, f) to $I = \ker f$. The fiber over a B-module M is identified with the A-extensions of B by M. This category will be denoted $\mathbf{Ex}_A(B, M)$.

Lemma 1.1.20. *$\mathbf{Ex}_A(B, M)$ is a groupoid.*

Proof. It is a straightforward application of the snake lemma. □

We will denote the set of connected components of $\mathbf{Ex}_A(B, M)$ by $\mathrm{Ex}_A(B, M)$. We want to show that this set has a natural structure of B-module and that it fits naturally into a long exact sequence. One can approach this question using a construction similar to that of Yoneda Ext module in homological algebra (see e.g. [Ser06, Ch. 1.1.2]); however, we prefer the approach via the cotangent complex. We view M as a complex concentrated in degree 0 and we think of $\mathbb{L}_{B/A}$ as a complex via Dold-Kan correspondence. We obtain in this way a cohomological complex

$$\mathrm{Hom}_B(\mathbb{L}_{B/A}, M) \simeq \mathrm{Der}_A(\perp_* B, M)$$

Its n-th cohomology is denoted as usual

$$\mathrm{Ext}^n_B(\mathbb{L}_{B/A}, M)$$

and it is called the André-Quillen cohomology of B with values in M.

Proposition 1.1.21. *We have a bijection between $\mathrm{Ext}^1_B(\mathbb{L}_{B/A}, M)$ and $\mathrm{Ex}_A(B, M)$.*

Proof. See [Ill71, Theorem 1.2.3]. □

Finally, if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of B-modules we obtain a long exact sequence

$$0 \to \mathrm{Der}_A(B, M') \to \mathrm{Der}_A(B, M) \to \mathrm{Der}_A(B, M'') \to \mathrm{Ex}_A(B, M') \to \cdots$$

1.2 Infinitesimal deformations

The goal of this section is to establish two of the main constructions of (formal) deformation theory, namely the construction of the Kodaira-Spencer map and the obstruction class of an infinitesimal deformation with respect to a small extension. More precisely, we begin by giving the basic definitions and analyzing the affine case. The first main result will be the rigidity of smooth affine schemes. Successively, we will analyze first order deformations, and we will be mainly concerned with the explicit construction of the Kodaira-Spencer map using Čech cohomology. Finally, we shall approach the problem of obstructions, which will be studied more in detail in the next section.

1.2.1 First definitions

We will work over a fixed field k. For the moment, we won't do any assumption on the characteristic. All the schemes we are considering are supposed to be schemes over k.

We can give a general definition of deformation of a given scheme X:

Definition 1.2.1. A deformation of a scheme X is a pullback diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d, "\pi"] \\ \mathrm{Spec}(k) \arrow[r, "s"] & S \end{tikzcd}$$

where $\pi\colon \mathfrak{X} \to S$ is a flat and surjective morphism of schemes. We will denote by $\xi := (\mathfrak{X}, S, s, \pi)$ such a deformation of X. We will also say that ξ is a deformation of X over the base (S, s).

Definition 1.2.2. Let X be a fixed scheme and let $\xi := (\mathfrak{X}, S, s, \pi)$ and $\xi' := (\mathfrak{X}', S, s, \pi')$ be two deformation of X over the same base (S, s). A morphism from ξ to ξ' is a morphism of schemes $f \colon \mathfrak{X} \to \mathfrak{X}'$ such that the diagram

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ \mathfrak{X} \arrow[rr, "f"] \arrow[dr, "\pi"'] & & \mathfrak{X}' \arrow[dl, "\pi'"] \\ & S & \end{tikzcd}$$

is commutative.

Remark 1.2.3. Fix a scheme X and consider a k-pointed scheme (S, s) (i.e. a scheme over $\mathrm{Spec}(k)$ with a fixed k-rational point). Then the deformations of X over the base (S, s) do form a category, where morphisms of deformations are the ones defined above. The composition is defined in the obvious way, and it is easily checked that the axioms of category are satisfied. More generally, one could consider deformations of X over a varying base scheme S. The result will be a category fibered in groupoids, and one could ask if this category form a stack.

Let X and S be schemes, and let $s \colon \mathrm{Spec}(k) \to S$ be a k-rational point on S. It is always possible to construct a deformation of X parametrized by S: namely, it is sufficient to consider the product

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \times_k S \arrow[d, "p_2"] \\ \mathrm{Spec}(k) \arrow[r, "s"] & S \end{tikzcd} \tag{1.2}$$

where the map f is induced by the maps

$$\begin{tikzcd} & X \\ X \arrow[ur, bend left, "\mathrm{id}_X"] \arrow[r, "f"] \arrow[dr, bend right, "s \circ h"'] & X \times_k S \arrow[u, "p_1"'] \arrow[d, "p_2"] \\ & S \end{tikzcd}$$

Observe that p_2 is flat and surjective because those properties are stable under pullback.

Definition 1.2.4. A deformation $(\mathfrak{X}, S, s, \pi)$ of a scheme X is said to be *trivial* if it is isomorphic to the product deformation (1.2).

Remark 1.2.5. As usual in this kind of situations, we would like to have an algebraic criterion to decide whether a deformation is trivial or not. This will be accomplished in the next part of this section, via the introduction of the *Kodaira-Spencer class map*.

In this mémoire we will be mainly concerned with *formal* deformation theory. This means that we won't consider deformations over arbitrary base schemes, but only over spectra of artinian rings. As we will explain better later, this corresponds to study formal neighborhood of a point in a would-be moduli space for a given moduli problem. Even though the moduli problem is not representable, we might be able to obtain information about this "virtual neighborhood", and use this to understand better the problem itself. For the moment, we introduce the notion of infinitesimal deformation:

Definition 1.2.6. A deformation $(\mathfrak{X}, S, s, \pi)$ is said to be *local* if $S = \mathrm{Spec}(A)$ for some commutative ring A; the deformation is said to be *infinitesimal* if $S = \mathrm{Spec}(A)$ with A a local artinian ring. If in particular $A = k[\varepsilon]$ (the ring of dual numbers), then the deformation is said to be a *first order deformation*.

Remark 1.2.7. Accordingly to Definition 1.2.1, if $(\mathfrak{S}, \mathrm{Spec}(A), s, \pi)$ is an infinitesimal deformation of X, then the local artinian ring A has necessarily residue field isomorphic to k.

Lemma 1.2.8. *Let $(\mathfrak{X}, \mathrm{Spec}(A), s, \pi)$ be an infinitesimal deformation of a given scheme X. The natural map $f\colon X \to \mathfrak{X}$ is a homeomorphism of topological spaces.*

Proof. It is well known that if $i\colon Y \hookrightarrow Z$ is a nilpotent closed immersion (i.e. a closed immersion defined by a nilpotent sheaf of ideals), then i^{top} is a homeomorphism of topological spaces. Since $s\colon \mathrm{Spec}(k) \to \mathrm{Spec}(A)$ satisfies this hypothesis, it will be sufficient to show that nilpotent closed immersions are stable under pullbacks, and this is completely straightforward: since the sheaf of ideals associated to a closed immersion is defined locally, and since nilpotence of a sheaf of ideals is a local property, we are readily reduced to the affine case. Now, if I is a nilpotent ideal of A, it is clear that for every A-algebra $A \to B$, the ideal IB is nilpotent, completing the proof. □

1.2.2 Infinitesimal deformations of affine schemes

We turn now to the proof of the first main result: the rigidity of smooth affine schemes. First of all we should define the notion of rigidity:

Definition 1.2.9. A scheme X is said to be *rigid* if every infinitesimal deformation $\xi = (\mathfrak{X}, S, s, \pi)$ of X is trivial.

Before starting the discussion, we need to give also another definition:

Definition 1.2.10. Let A be a k-algebra and let $\varphi\colon A' \to A$ be a square-zero k-extension of A. We say that the extension is *small* if $\ker\varphi \simeq k$, then the extension is said to be *small*.

We will return on the notion of (small) extension in the next section.

Lemma 1.2.11. *Let $Z_0 \subset Z$ be a closed immersion defined by a sheaf of nilpotent ideals. Then Z is affine if and only if Z_0 is affine.*

Proof. It is a well known fact that a closed subscheme of an affine scheme is again affine. For the other direction, see [EGAI, Proposition 5.1.9]. □

Let B_0 be a k-algebra and let $X_0 := \mathrm{Spec}(B_0)$. If

$$\begin{tikzcd} X_0 \arrow[r, hook, "i"] \arrow[d] & \mathfrak{X} \arrow[d, "\pi"] \\ \mathrm{Spec}(k) \arrow[r, "s"] & \mathrm{Spec}(A) \end{tikzcd}$$

is an infinitesimal deformation of X_0, then $i\colon X_0 \to \mathfrak{X}$ is a closed immersion defined by a sheaf of ideals $\mathcal{J}$. It is straightforward to check locally that $\mathcal{J}$ is a sheaf of nilpotent ideals. It follows from Lemma 1.2.11 that $\mathfrak{X}$ has to be an affine scheme, that is $\mathfrak{X} = \mathrm{Spec}(B)$. Therefore we are immediately reduced to study pushout diagrams

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & k \arrow[d] \\ B \arrow[r] & B_0 \end{tikzcd}$$

where the map f is flat. We will say that the A-algebra B is an A-deformation of B_0.

If B and B' are A-deformation of B_0, then Definition 1.2.2 translates in this context as follows: a morphism from B to B' is an A-algebra morphism $\varphi\colon B \to B'$ making the following diagram to commute:

$$\begin{tikzcd} & B_0 & \\ B \arrow[ur] \arrow[rr, "\varphi"] & & B' \arrow[ul] \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Lemma 1.2.12. *If $\varphi\colon B \to B'$ is a morphism of deformations of B_0 over A, then φ is an isomorphism.*

Proof. Let $\mathfrak{m}_A$ be the maximal ideal of A. Regarding B and B' as A-modules, let $K :=$ coker φ. Tensoring

$$B \to B' \to K \to 0$$

with $\kappa(\mathfrak{m}_A) = A/\mathfrak{m}_A$ we obtain $K \otimes_A \kappa(\mathfrak{m}_A) = 0$. Since $\mathfrak{m}_A$ is nilpotent, Nakayama implies $K = 0$. Let now $I = \ker \varphi$. Tensoring

$$0 \to I \to B \to B' \to 0$$

with $\kappa(\mathfrak{m}_A)$ we obtain

$$\mathrm{Tor}_1^A(\kappa(\mathfrak{m}_A), B') \to I \otimes_A \kappa(\mathfrak{m}_A) \to B \otimes_A \kappa(\mathfrak{m}_A) \to B' \otimes_A \kappa(\mathfrak{m}_A) \to 0$$

Since B' is flat over A, the first term vanishes and since $B \otimes_A \kappa(\mathfrak{m}_A) \to B' \otimes_A \kappa(\mathfrak{m}_A)$ is an isomorphism, we see that $I \otimes_A \kappa(\mathfrak{m}_A) = 0$, i.e. $I = 0$. □

Lemma 1.2.13. *Let A be an artinian local k-algebra with residue field isomorphic to k. If $\dim_k(A) = 2$ then there exists a* non-canonical *isomorphism $A \simeq k[\varepsilon]$, where $k[\varepsilon] := k[X]/(X^2)$ is the ring of dual numbers.*

Proof. Let $\mathfrak{m}_A$ be the maximal ideal of A. Since $\dim_k(A) = 2$, it follows necessarily that

$$\dim_k \mathfrak{m}_A/\mathfrak{m}_A^2 \leq 1$$

so that $\mathfrak{m}_A$ is principal by Nakayama's Lemma. Let $\mathfrak{m}_A = (t)$, with $t \in A$. Then 1_A and t are k-linearly independent, so that

$$A \simeq k \oplus k \cdot t$$

Finally, we must have $t^2 = 0$ because otherwise $1, t, t^2$ would be k-linearly independent. Therefore, the canonical morphism $k[\varepsilon] \to A$ defined by $\varepsilon \mapsto t$ is an isomorphism. □

Theorem 1.2.14. *Every smooth k-algebra is rigid.*

Proof. Consider an infinitesimal deformation

$$\begin{tikzcd} B \arrow[r] & B_0 \\ A \arrow[u, "f"] \arrow[r] & k \arrow[u] \end{tikzcd}$$

with $f\colon A \to B$ flat. We will proceed by induction on $d = \dim_k(A)$. If $\dim_k(A) = 1$, $A = k$ and there is nothing to prove. If $\dim_k(A) = 2$, apply Lemma 1.2.13 to conclude that $A \simeq \kappa[\varepsilon]$.

In this case, the fiber of $\operatorname{Spec} B \to \operatorname{Spec} k[\varepsilon]$ is smooth by hypothesis, so that $k[\varepsilon] \to B$ is smooth. Introduce the trivial deformation

$$B_0[\varepsilon] := B_0 \otimes_k k[\varepsilon]$$

and consider the commutative diagram

$$\begin{tikzcd} B \arrow[r] \arrow[dr, dotted, "g"] & B_0 \\ k[\varepsilon] \arrow[u] \arrow[r] & B_0[\varepsilon] \arrow[u] \end{tikzcd}$$

The kernel of $B_0[\varepsilon] \to B_0$ is generated by $\varepsilon \in B_0[\varepsilon]$ and so it is square-zero. It follows from smoothness of $k[\varepsilon] \to B$ that the dotted arrow exists. Lemma 1.2.12 implies that g is an isomorphism.

Now assume the statement for $(d-1)$ and let $\dim_k(A) = d \geq 2$. Choose an element $t \in \mathfrak{m}_A$, satisfying $t \neq 0$ and $t^2 = 0$ and set $A' := A/(t)$, so that $\dim_k(A') = d-1$. Therefore we have a small extension:

$$0 \to (t) \to A \to A' \to 0$$

Consider the commutative diagram

$$\begin{tikzcd} B \arrow[r] \arrow[dr, dotted, "g"] & B \otimes_A A' \\ A \arrow[u, "f"] \arrow[r] & B_0 \otimes_k A \arrow[u] \end{tikzcd} \tag{1.3}$$

Since

$$(B \otimes_A A') \otimes_{A'} k = B \otimes_A k = B_0$$

we deduce that $A' \to B \otimes_A A'$ is a deformation of B_0 over A'; induction hypothesis implies the existence of an isomorphism $B \otimes_A A' \simeq B_0 \otimes_k A'$. Since the kernel of

$$B_0 \otimes_k A \to B_0 \otimes_K A'$$

is square-zero, the smoothness of $f\colon A \to B$ implies the existence of the dotted arrow in diagram (1.3). Lemma 1.2.12 implies that g is an isomorphism, completing the proof. □

1.2.3 Locally trivial deformations

Our next goal is to give a first construction of the Kodaira-Spencer map, which allows to give a geometric interpretation to the first cohomology group $H^1(X, \mathcal{T}_X)$, where $\mathcal{T}_X$ is the tangent sheaf of the algebraic variety X. We begin with a discussion about locally trivial deformations.

Lemma 1.2.15. *Let $\xi = (\mathfrak{X}, S, s, \pi)$ be any deformation of a scheme X. The natural map $X \to \mathfrak{X}$ is affine.*

Proof. Affine morphisms are stable under pullback, and $\mathrm{Spec}(k) \to S$ is clearly affine. □

Lemma 1.2.16. *Let $\xi = (\mathfrak{X}, S, s, \pi)$ be an infinitesimal deformation of a scheme X. Let $i\colon U \subset \mathfrak{X}$ be an open immersion with $U \neq \emptyset$ and let $V := U \times_{\mathfrak{X}} X \subset X$ be the corresponding open immersion on X. Then the diagram*

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d, "\pi \circ i"] \\ \mathrm{Spec}(k) \arrow[r] & S \end{tikzcd}$$

expresses U as a deformation of V.

Proof. Easy abstract nonsense shows that the diagram is a pullback. Let $x \in U$ be any point; then $\pi(x)$ has to be the only point of S; it follows that $\pi \circ i$ is surjective. Finally, $\pi \circ i$ is flat because open immersions are flat and flat morphisms are stable under composition. □

Definition 1.2.17. Let $\xi = (\mathfrak{X}, S, s, \pi)$ be an infinitesimal deformation of a scheme X. We say that ξ is locally trivial if for each point $x \in X$ there is an open affine neighborhood $x \in U \subset \mathfrak{X}$ such that the induced deformation

$$\begin{tikzcd} U \times_{\mathfrak{X}} X \arrow[r] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & S \end{tikzcd}$$

is trivial.

1.3 Formal theory

We explained in the previous section what a general deformation is. To study the structure of a scheme X, we might be interested in analysing all the deformations of X over an arbitrary base scheme S. In this way, we are lead to the introduction of the category of deformations of X; the associated category fibred in groupoids will be denoted by $\mathbf{Def}(X)$. To simplify a bit the problem, we could consider as well the associated category fibred in sets, $\mathrm{Def}(X)$. This is nothing but a classical moduli problem.

More generally, one could consider other deformation problems: for example, he might be interested in the deformation of line bundles over a given scheme, or to deform a scheme as subscheme of a given scheme. We should develop a sufficiently flexible language in order to deal with all these needs. The simplest thing to do is to say that a deformation problem is a particular kind of moduli problem, that is a functor

$$X\colon \mathbf{Sch}_{/k,*} \to \mathbf{Set} \tag{1.4}$$

from the category of k-pointed schemes to $\mathbf{Set}$. We might require X to satisfy some additional property; for example we might require $X(k)$ to be a one point set.

Formulated in this way, it would be really hard to develop a satisfying enough formulation of deformation theory. Formal deformation theory is meant to provide a simplified approach to the subject; namely, assume that we are given a deformation problem (1.4) and choose a (k-rational) point $\eta \in X(k)$. We would like to analyse the "local structure" of X around the point η; to understand what we mean, assume that X is represented by a moduli space $\mathcal{M}$. Then the geometry of $\mathcal{M}$ could be rather complicated, but we could expect to have a good understanding of formal neighbourhoods of a given point $\eta \in \mathcal{M}$. The key observation to develop formal deformation theory is to observe that this formal neighbourhood can be entirely recovered from the moduli problem X, even without knowing the existence of $\mathcal{M}$. In fact, assume that such a formal neighbourhood is described by a complete local noetherian ring A, with maximal ideal $\mathfrak{m}_A$. Then every ring $A/\mathfrak{m}_A^n$ is artinian and

$$A \simeq \varprojlim_n A/\mathfrak{m}_A^n$$

This means that A is a pro-object in the category Art_k of local artinian k-algebras with residue field isomorphic to k. Moreover, A corepresents the functor obtained by X

$$\widehat{X}\colon \mathrm{Art}_k \to \mathbf{Set}$$

described by the formula

$$\widehat{X}(R) = X(\mathrm{Spec}(R)) \times_{X(k)} \{\eta\}$$

In general, the goal of formal deformation theory is to study "formal neighbourhoods" of special kind of moduli problems. We will formalize what we precisely mean with the notion of *deformation functor*.

1.3.1 Functor of Artin rings

Let k be a field. We will denote by Art_k the full subcategory of $\mathbf{CRing}$ having as objects local artinian k-algebras with residue field isomorphic to k. We will denote by $\widehat{\mathrm{Art}_k}$ (resp. Art_k^*) the full subcategory of $\mathbf{CRing}$ having as objects (complete) local noetherian k-algebras with residue field isomorphic to k.

Definition 1.3.1. A functor of artin rings is a (covariant) functor $F\colon \mathrm{Art}_k \to \mathbf{Set}$ such that $F(k)$ is a one-point set.

Example 1.3.2. Consider a moduli problem

$$F\colon \mathbf{CRing} \to \mathbf{Set}$$

If A is a commutative ring, any element of $F(A)$ is said to be an A-rational point of F. Fix a field k and let $\eta \in F(k)$ be a k-rational point for the moduli problem F. We obtain a functor of artin rings

$$\widehat{F}\colon \mathrm{Art}_k \to \mathbf{Set}$$

defining $\widehat{F}(A)$ to be the pullback of the diagram

$$\begin{tikzcd} F(A) \times_{F(k)} \{\eta\} \arrow[r] \arrow[d] & F(A) \arrow[d] \\ \{\eta\} \arrow[r] & F(k) \end{tikzcd}$$

This is called the completion of F at the point $\eta \in F(k)$.

Example 1.3.3. Suppose that the moduli problem $F\colon \mathbf{CRing} \to \mathbf{Set}$ of previous example is representable by a scheme X. Then a k-rational point of F is simply a k-point $x\colon \operatorname{Spec} k \to X$. The induced functor of artin rings can be therefore described as

$$\widehat{F}(A) = \operatorname{Hom}_{\mathbf{CRing}}(\widehat{\mathscr{O}_{X,x}}, A)$$

In fact, the universal property of completion shows that any morphism $f\colon \operatorname{Spec}(A) \to X$ with A local artinian k-algebra and with image x must be induced by a morphism from the completed ring $\widehat{\mathscr{O}_{X,x}}$.

Previous examples show how to associate to every moduli problem many functors of artin rings. Motivated from example 1.3.3 we will refer to such functors also as (classical) *formal moduli problems*. The same example show that it is unreasonable to ask for a formal moduli problem to be representable. However, it might be interesting to consider the following weaker property:

Definition 1.3.4. A functor of artin rings $F\colon \mathrm{Art}_\Lambda \to \mathbf{Set}$ is said to be *prorepresentable* if it is a the restriction of the representable functor h_R to Art_k for some complete local Λ-algebra R.

Lemma 1.3.5. *The category* Art_k *has pullbacks.*

Proof. It will be sufficient to show that if in a diagram

$$\begin{tikzcd} B \times_A C \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & A \end{tikzcd}$$

the rings A, B and C are in Art_k, then so is $B \times_A C$. Observe first of all that the product of artinian rings is again artinian: in fact

$$\operatorname{Spec}(B \times C) = \operatorname{Spec}(B) \sqcup \operatorname{Spec}(C)$$

so that $\dim.\mathrm{Krull}(B \times C) = 0$. Moreover, $B \times C$ satisfies clearly the ascending chain condition on ideals, so that it is noetherian; it follows that $B \times C$ is an artinian k-algebras whose maximal points are k-rationals. The structure theorem for artinian rings readily implies $\dim_k B \times C < \infty$. Since

$$\dim_k B \times_A C \leq \dim_k B \times C < \infty$$

it follows that $B \times_A C$ satisfies the descending chain condition on ideals, i.e. it is artinian as well. $\square$

Let now

$$\begin{tikzcd} B \times_A C \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & A \end{tikzcd}$$

be a cartesian diagram in Art_k. Let

$$\alpha\colon F(B \times_A C) \to F(B) \times_{F(A)} F(C) \tag{1.5}$$

be the natural map.

Definition 1.3.6. A functor of Artin rings $F\colon \mathrm{Art}_k \to \mathbf{Set}$ is said to be a *deformation functor* if:

1. the map α appearing in (1.5) is surjective whenever $B \to A$ is a small extension;
2. the map α appearing in (1.5) is an isomorphism whenever $A = k$.

Proposition 1.3.7. *Let $F = h_R$ be a prorepresentable functor. Then F satisfies the following properties:*

1. *$F(k)$ consists of one element;*
2. *F commutes with pullbacks;*
3. *$F(k[\varepsilon])$ is a finite dimensional k-vector space.*

In particular F is a deformation functor.

Proof. The statements 1. and 2. are obvious. For the last one,

$$F(k[\varepsilon]) = \mathrm{Hom}_\Lambda(R, k[\varepsilon]) = \mathrm{Der}_\Lambda(R, k)$$

is the relative tangent space of R. □

Schlessinger's criterion gives a characterization of those functors of Artin rings which are representable.

Theorem 1.3.8 (Schlessinger)**.** *Let $F\colon \mathrm{Art}_k \to \mathbf{Set}$ be a functor of Artin rings. Given a pullback in Art_k*

$$\begin{tikzcd} B \times_A C \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & A \end{tikzcd}$$

let

$$\alpha\colon F(B \times_A C) \to F(B) \times_{F(A)} F(C)$$

be the natural map. Then F is prorepresentable if and only if it satisfies the following conditions:

1. *the map α is bijective whenever $B = C$ and $B \to A$ is a small extension;*
2. *if $C \to A$ is a small extension, then α is surjective;*
3. *if $A = k$ and $C = k[\varepsilon]/(\varepsilon^2)$ the map α is bijective;*
4. $\dim_k(T_F) < \infty$.

Proof. See [Ser06, Theorem 2.3.2]. □

Chapter 2

Higher algebra

The purpose of this chapter is to develop enough language to deal with Derived Algebraic Geometry. A higher categorical perspective is essential in order to do so; however, we don't really need a complete theory of Higher Category Theory: it will be more than enough to be able to deal with $(\infty, 1)$-categories. There is an axiomatic approach to the theory of $(\infty, 1)$-categories, as well as several models for it. It would be indeed really useful to know and master them all; despite that, in writing this mémore we made a choice: we choose to follow J. Lurie and to use the model given by quasicategories.

We organized the chapter in three part. The first one is essentially a compendium of the needed results of [HTT]; the general goal is to develop $(\infty, 1)$-categorical analogues of the main constructions and results of classical category theory (overcategories, limits and colimits, adjunctions). Since we are working with the language of quasicategories, it is unavoidable to deal with simplicial and combinatorial arguments; we grouped these technical details in special sections, called "interludes".

In the second part of the chapter, we focus on the relationship between quasicategories and model categories. In particular, we exhibit a way of passing from a model category to a quasicategory, via the Dwyer-Kan localization. The main reference for this part is the article [DK80]; we completed several details left to the reader in that article.

Finally, the last part is devoted to the basics of higher algebra. We followed closely the first sections of [HA, Chapter 1]. We give the definition of stable $(\infty, 1)$-category and we construct loop and suspension functors, proving the canonical adjunction between them. We conclude introducing the notion of spectrum in a quasicategory.

2.1 The language of quasicategories

In this first section we introduce the language of quasicategories. This notion has been developed systematically by A. Joyal in [Joy08] and J. Lurie made an extensive use of it in all his works up to this moment, starting with his PhD thesis and with the book [HTT]. Among other things, this theory gives a satisfying model for the theory of $(\infty, 1)$-categories. However, we won't try to give motivations for this theory too seriously; the book [HTT] fulfill this task in an excellent way.

The reader is supposed to be familiar with the language of simplicial sets. Appendix A is meant to give a quick overview of all the needed notions. In particular, in this section we will use the join of simplicial sets without recalling its construction.

2.1.1 First definitions

Definition 2.1.1. An quasicategory (or ∞-category, or $(\infty, 1)$-category) is a simplicial set $S \in$ **sSet** satisfying the weak Kan condition: for any $n \in N$, any $0 < i < n$ and any morphism $\varphi\colon \Lambda^n_i \to S$ there is a commutative diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\varphi"] \arrow[d] & S \\ \Delta^n \arrow[ur, "\psi"'] \end{tikzcd} \tag{2.1}$$

Example 2.1.2. Let $\mathcal{C}$ be an ordinary category. Then the simplicial set $\mathrm{N}\mathcal{C}$ is a quasicategory. In fact, we can characterize the essential image of the nerve functor $N\colon \mathbf{Cat} \to \mathbf{sSet}$: it consists exactly of those simplicial sets K such that for each $n \in N$, each $0 < i < n$ and any morphism $\varphi\colon \Lambda^n_i \to S$ there exists a *unique* morphism $\psi\colon \Delta^n \to S$ making the diagram (2.1) commutative. This is sometimes called the *Segal condition.*

Example 2.1.3. Let $X \in \mathbf{CGHaus}$. Then the singular complex $\mathrm{Sing}(X) \in \mathbf{sSet}$ is a quasicategory: in fact, it is a Kan complex. Conversely, every Kan complex is weakly equivalent to a simplicial set arising in this way. Sometimes, we will refer to Kan complexes as ∞-groupoids.

The following definition are quite expected:

Definition 2.1.4. Let S be a quasicategory. An object of S is a vertex of S, i.e. a morphism $\Delta^0 \to S$; an edge of S is a 1-simplex of S, i.e. a morphism $\Delta^1 \to S$. We will denote by $\mathrm{Ob}(S)$ the set of objects of S.

If S is a quasicategory and X, Y are objects of S, then an edge $e\colon \Delta^1 \to S$ is said to be an arrow from X to Y if

$$e \circ d^1 = X, \qquad e \circ d^0 = Y$$

We will see that going deeper in the framework of quasicategories, this notion of arrows matches perfectly the idea that one has from (intuitive) higher category theory.

Notation 2.1.5. If $X, Y \in \mathrm{Ob}(S)$, we will denote by $\hom_S(X, Y)$ the set of arrows from X to Y.

We can use the few ideas introduced up to this moment and the classical homotopy theory for simplicial sets to build the homotopy category of a quasicategory. We begin with the following remark:

Lemma 2.1.6. *Let S be a quasicategory and let $f_0\colon \partial\Delta^n \to S$ be any morphism. The homotopy relation* $\mathrm{rel}.\partial\Delta^n$ *defines an equivalence on the set S_{f_0} of n-simplexes of S extending f_0.*

Proof. Let $f \in S_{f_0}$. Then $s_n(f)\colon \Delta^{n+1} \to S$ and

$$d_n s_n(f) = d_{n+1} s_n(f) = f$$

while for $0 \le i < n$ we have:

$$d_i s_n(f) = s_{n-1} d_i(f)$$

so that the relation is reflexive.

Assume now that $f, g, h \in S_{f_0}$ and let $\varphi\colon \Delta^{n+1} \to S$ be a homotopy from f to g, $\psi\colon \Delta^{n+1} \to S$ be a homotopy from h to g. Consider the $n+2$ $(n+1)$–simplexes

$$d_0 s_n s_n f, \dots, d_{n-1} s_n s_n f, -, \varphi, \psi$$

Then we have

$$\begin{aligned} d_i d_j s_n s_n f &= d_{j-1} d_i s_n s_n f && i < j < n \\ d_i \varphi &= s_{n-1} d_i f = d_n s_{n-1} s_{n-1} d_i f && i < n \\ &= d_n d_i s_n s_n f \\ d_i \psi &= s_{n-1} d_i g = d_n s_{n-1} s_{n-1} d_i g && i < n \\ &= d_n s_{n-1} s_{n-1} d_i f = d_n d_i s_n s_n f \\ d_{n+1} \psi &= g = d_{n+1} \varphi \end{aligned}$$

It follows that we have a morphism

$$\sigma\colon \Lambda^{n+2}_n \to S$$

Since S is a quasicategory we can extend it to a morphism $\omega\colon \Delta^{n+2} \to S$, and $d_n\omega$ defines a homotopy from f to h:

$$\begin{aligned} d_n d_n \omega &= d_n d_{n+1}\omega = d_n\varphi = f \\ d_{n+1} d_n \omega &= d_n d_{n+2}\omega = d_n\psi = h \end{aligned}$$

Now we can show that the homotopy relation is symmetric: if ω realizes a homotopy from α to β, apply the previous argument taking $f = g = \beta$, $h = \alpha$, $\varphi = s_n(g)$, $\psi = \omega$.

Finally, the same argument and the symmetry show that the homotopy relation is also transitive. □

Remark 2.1.7. The proof is slightly trickier than in the classical context, where S is assumed to be a Kan complex. In [May69] the statement is proved in this case, using the horn Λ^{n+2}_{n+2} which allows a more direct proof.

Using Lemma 2.1.6 we can move the next step toward the definition of the homotopy category of a quasicategory S. Let $\mathrm{h}S$ be the graph whose vertexes are the objects of S. If $X, Y \in \mathrm{Ob}(S)$, we set

$$\mathrm{Hom}_{\mathrm{h}S}(X,Y) := \mathrm{hom}_S(X,Y)/\sim$$

where $\sim$ is the homotopy relation of 1-simplexes relative to $\partial\Delta^1$. To define the composition, proceed as follows: if $[\alpha] \in \mathrm{Hom}_{\mathrm{h}S}(X,Y)$ and $[\beta] \in \mathrm{Hom}_{\mathrm{h}S}(Y,Z)$ are homotopy classes represented by α and β we observe that they determine a map

$$\Lambda^2_1 \to S$$

which can be extended to a 2-simplex $\omega\colon \Delta^2 \to S$. We set

$$[\beta]\circ[\alpha] := [d_1\omega]$$

Lemma 2.1.8. *The above definition depend only on the homotopy class of α and β.*

Proof. Assume that $\sigma\colon \Delta^2 \to S$ is another 2-simplex completing the horn inclusion

$$(\alpha, -, \beta)\colon \Lambda^2_1 \to S$$

Then we can consider the map

$$(s_1(\beta), -, \sigma, \omega)\colon \Lambda^3_1 \to S$$

Completing it into a map $\tau\colon \Delta^3 \to S$ we obtain

$$\begin{aligned} d_0 d_1 \tau &= d_0 d_0 \tau = d_0 s_1(\beta) = s_0(Z) \\ d_1 d_1 \tau &= d_1 d_2 \tau = d_1\sigma \\ d_2 d_1 \tau &= d_1 d_3 \tau = d_1\omega \end{aligned}$$

showing that $d_1\tau$ is a homotopy between $d_1\sigma$ and $d_1\omega$.

In a similar way it is seen that the definition does not depend on the choice of the representatives α and β (cfr. [HTT, Proposition 1.2.3.7] for the details). □

Theorem 2.1.9. *The construction above yields a category* $\mathrm{h}S$*. Moreover, if S is a Kan complex,* $\mathrm{h}S$ *is a groupoid.*

Proof. The proof is entirely straightforward. We check the existence of the identities, but we refer to [HTT, Proposition 1.2.3.8] for the associativity of the composition.

If x is an object of $\mathrm{h}S$, define id_x to be $s_0(x) \in S_1$. If $f\colon x \to y$ represents a morphism in $\mathrm{h}S$, we observe that $s_0(f)$ is a 2-simplex satisfying

$$d_0 s_0(f) = d_1 s_0(f) = f, \qquad d_2 s_0(f) = s_0 d_1(f) = s_0(x)$$

showing that $[f] \circ \mathrm{id}_x = [f]$. A similar argument shows that $\mathrm{id}_x \circ [g] = [g]$ for every morphism $[g]: z \to x$.

Finally, if S is a Kan complex and $[f]: x \to y$ is a morphism in hS, complete the horn inclusion $\Lambda^2_2 \to S$ depicted as

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[rr, "s_0(x)"'] & & x \end{tikzcd}$$

to obtain a left inverse for f; using the horn Λ^2_0 one sees that f has also a right inverse, so that it is invertible. □

Definition 2.1.10. Let S be a quasicategory. The category hS built in Theorem 2.1.9 will be called the *homotopy category* of S.

Remark 2.1.11. We will see later that the homotopy category of a quasicategory can be considered as a simplicial category. This will allow to introduce the correct definition of equivalence of quasicategories.

Definition 2.1.12. Let S be a quasicategory. An edge $\phi: \Delta^1 \to S$ is said to be an equivalence if it is an isomorphism in hS.

Finally, we introduce the notion of full subcategory. Let S be a simplicial set and let V_0 be a set of vertexes of S. We define a new sub-simplicial set S_0 of S by saying that an n-simplex $\sigma: \Delta^n \to S$ lies in S_0 if and only if every composition

$$\Delta^0 \to \Delta^n \to S$$

factors through V_0. It is straightforward to see that S_0 is a simplicial set. Moreover, if S was a quasicategory to begin with, then S_0 is a quasicategory: in fact, the definition itself of S_0 shows that in a diagram

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d, tail, "\varphi"] & S_0 \arrow[r, tail] & S \\ \Delta^n \arrow[urr, "\sigma"'] & & \end{tikzcd}$$

the n-simplex σ belongs to S_0. In fact every map

$$\Delta^0 \to \Delta^n \xrightarrow{\sigma} S$$

factors as

$$\Delta^0 \to \Lambda^n_j \xrightarrow{\varphi} S_0$$

so that we are done. We give the following definition:

Definition 2.1.13. Let S be a quasicategory and let V_0 be a subset of its vertexes. The category S_0 defined above is said to be the full subcategory of S spanned by V_0.

Remark 2.1.14. From now on we won't make any distinction between the words "quasicategory", "$(\infty, 1)$-category" and "∞-category".

2.1.2 Interlude I: fibrations of simplicial sets

Before going on any further our framework, we will need to develop a number of techniques in order to deal with quasicategories. In fact, the proofs are usually hard without the correct machinery, and it is quite difficult to proceed in the theory avoiding such technicalities. The purpose of this part is to describe the formalism of fibrations and anodyne extensions in the context of quasicategories. In order to reduce the number of technical arguments presented here, we moved some of them into the Appendix; a proper reference is given whenever needed.

We begin with a definition:

Definition 2.1.15. A morphism $f\colon K \to L$ of simplicial sets is said to be:

1. a *left fibration* if it has the RLP with respect to every horn inclusion $\Lambda^n_i \to \Delta^n$ for $0 \le i < n$;
2. a *right fibration* if it has the RLP with respect to every horn inclusion $\Lambda^n_i \to \Delta^n$ for $0 < i \le n$;
3. a *inner fbration* if it has the RLP with respect to every horn inclusion $\Lambda^n_i \to \Delta^n$ for $0 < i < n$;
4. *left anodyne* if it has the LLP with respect to every left fibration;
5. *right anodyne* if it has the LLP with respect to every right fibration;
6. *inner anodyne* if it has the LLP with respect to every inner fibration.

Left fibrations and left anodyne extensions

Throughout this chapter it will appear more and more clear that left fibrations are the correct generalization in the ∞-categorical setting of the notion of category cofibered in groupoids. For the moment, we will simply give equivalent characterization of left anodyne extensions in order to obtain some stability property. We will apply this theory to show that if $p\colon X \to S$ is a left fibration and $f\colon \Delta^1 \to S$ is an edge in S from s to s', then there exists a functor $X_s \to X_{s'}$, which is well defined up to homotopy.

The characterization of left anodyne extensions consists in describing this class of maps as the saturation of certain well understood families of maps. The resulting theorem, due to Joyal, is the analogue of the well-known theorem of Gabriel and Zisman concerning Kan fibrations of simplicial sets. We will omit the proof, but we will analyze some of its consequences:

Theorem 2.1.16. *The following collections of morphisms generate the same weakly saturated class of morphisms of* **sSet***:*

1. *the collection A_1 of all horn inclusions $\Lambda^n_i \subseteq \Delta^n$ for $0 \le i < n$;*
2. *the collection A_2 of all inclusions*
$$(\Delta^m \times \{0\}) \coprod_{\partial\Delta^m \times \{0\}} (\partial\Delta^m \times \Delta^1) \subseteq \Delta^m \times \Delta^1$$
3. *the collection A_3 of all inclusions*
$$(S' \times \{0\}) \coprod_{S \times \{0\}} (S \times \Delta^1) \subseteq S' \times \Delta^1$$
where $S \subseteq S'$.

Proof. See [HTT, Proposition 2.1.2.6]. □

Corollary 2.1.17. *Let $i\colon A \to A'$ be a left anodyne map and let $j\colon B \to B'$ be a (standard) cofibration. Then the induced map*

$$\widetilde{F}(i,j)\colon (A \times B') \coprod_{A \times B} (A' \times B) \to A' \times B' \tag{2.2}$$

is left anodyne.

Proof. Fix the cofibration $j\colon B \to B'$ and let $g\colon \emptyset \to B'$ be the unique map; Lemma A.2.9 shows that the hypothesis of Proposition A.2.8 are satisfied if we take $F = - \times B'$, $\mathcal{S}_1 = \{g\}$ and $\mathcal{S}_2$ equal to the class of left anodyne extensions. It follows that the class $\mathcal{M}$ of maps

$A \to A'$ such that $A \times B' \to A' \times B'$ is left anodyne is saturated; using Theorem 2.1.16 we see that if we show that $\mathcal{M}$ contains the set A_3 of the inclusions of the form

$$\alpha\colon (S' \times \{0\}) \coprod_{S \times \{0\}} (S \times \Delta^1) \subseteq S' \times \Delta^1$$

for $S \subseteq S'$, then $\mathcal{M}$ will contain all the left anodyne extensions. However, A_3 is stable under the application of $- \times B'$ (because $S \times B' \subseteq S' \times B'$), so that $A_3 \subset \mathcal{M}$. Now apply again Lemma A.2.9 and Proposition A.2.8 to the set $\mathcal{S}_1 = \{g, j\}$: we obtain another saturated set $\mathcal{M}'$ which is defined as the class of maps $i\colon A \to A'$ such that $\widetilde{F}(i,g)$ and $\widetilde{F}(i,j)$ are left anodyne. We already showed that $\widetilde{F}(i,g)$ is left anodyne for every left anodyne map i; to show that $\mathcal{M}'$ contains all the left anodyne maps, we are left to check that $\widetilde{F}(\alpha, j)$ is left anodyne for every $\alpha \in A_3$. Nonsense shows that this is true (cfr. the proof of [GoJa, Corollary I.4.6]). □

Let $p\colon X \to S$ be a left fibration of simplicial sets. For every edge $f\colon s \to s'$ in S, we can consider the following diagram

$$\begin{tikzcd} \{0\} \times X_s \arrow[r] \arrow[d] & & X \arrow[d, "p"] \\ \Delta^1 \times X_s \arrow[r] \arrow[urr, dotted] & \Delta^1 \arrow[r, "f"] & S \end{tikzcd}$$

Previous Corollary shows that $\{0\} \times X_s \to \Delta^1 \times X_s$ is left anodyne, so that the lifting problem has solution. We therefore obtain a map

$$f_!\colon X_s \simeq \{1\} \times X_s \to X_{s'} \subset X$$

Inner fibrations and inner anodyne extensions

An analogue of Theorem 2.1.16 holds for inner anodyne extensions. We now focus on this class of maps and we develop some stability property.

Theorem 2.1.18. *The following collections of morphisms generate the same saturated class of morphisms in* **sSet***:*

1. *the collection of all horn inclusions $\Lambda^n_i \subset \Delta^n$, $0 < i < n$;*
2. *the collection of all inclusions*
$$(\Delta^m \times \Lambda^2_1) \coprod_{\partial\Delta^m \times \Lambda^2_1} (\partial\Delta^m \times \Delta^2) \subseteq \Delta^m \times \Delta^2$$
3. *the collection of all inclusions*
$$(S' \times \Lambda^2_1) \coprod_{S \times \Lambda^2_1} (S \times \Delta^2) \subseteq S' \times \Delta^2$$
where $S \subseteq S'$.

Proof. See [HTT, Proposition 2.3.2.1]. □

Corollary 2.1.19. *Let $i\colon A \to A'$ be an inner anodyne map of simplicial sets and let $j\colon B \to B'$ be a cofibration. Then*

$$(A \times B') \coprod_{A \times B} (A' \times B) \to A' \times B'$$

is inner anodyne.

Corollary 2.1.20. *Let $i\colon A \to A'$ be an inner anodyne extension and let K be a simplicial set. Then the induced map $A \times K \to A' \times K$ is inner anodyne.*

Proof. In Corollary 2.1.19 choose $B = \emptyset$, $B' = K$ and observe that $A \times \emptyset = A' \times \emptyset = \emptyset$. □

2.1.3 Overcategories and undercategories

Lemma 2.1.21. *Let $p\colon K \to S$ be a morphism of simplicial sets, where S is a quasicategory. There exists a simplicial set $S_{/p}$ satisfying the following universal property: for any other simplicial set Y we have a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{sSet}}(Y, S_{/p}) \simeq \mathrm{Hom}_p(Y \star K, S) \tag{2.3}$$

where the subscript p means that we are considering the subset of those maps $f\colon Y \star K \to S$ such that $f|_K \equiv p$.

Proof. Define

$$S_{/p} := \mathrm{Hom}_p(\Delta^\bullet \star K, S) \in \mathbf{sSet}$$

Then equation 2.3 hold by definition when $Y = \Delta^n$ for some $n \in \mathbb{N}$. The general formula follows from the fact that both sides commute with colimits in Y. □

Remark 2.1.22. Let $p\colon K \to S$ as above. From the explicit construction of $S_{/p}$ it follows that we have a natural "forgetful functor"

$$f\colon S_{/p} \to K$$

obtained applying $\mathrm{Hom}_{\mathbf{sSet}}(-, S)$ to the inclusion map

$$K \to \Delta^\bullet \star K$$

(on the left side, K is considered as constant bisimplicial set). In a similar way, considering the inclusion

$$\Delta^\bullet \to \Delta^\bullet \star K$$

we obtain a natural map

$$S_{/p} \to \mathrm{Hom}_{\mathbf{sSet}}(\Delta^\bullet, S) \simeq S$$

which is the other natural forgetful functor.

The idea is to use this simplicial set $S_{/p}$ as the ∞-categorical analogue of a comma category. However, before taking it as a definition, we would like to know whether $S_{/p}$ is an ∞-category or not. The answer is affirmative, and using the machinery of left fibrations it is not too hard to prove it.

Lemma 2.1.23. *Let $f\colon A_0 \subseteq A$ and $g\colon B_0 \subseteq B$ be inclusions of simplicial sets. If f is right anodyne or g is left anodyne then the map*

$$h = h(f,g)\colon (A_0 \star B) \coprod_{A_0 \star B_0} (A \star B_0) \subseteq A \star B$$

is inner anodyne. If f is left anodyne then h is left anodyne.

Proof. In the notations of Proposition A.2.6, choose the bifunctor F to be $- \star -$; choose $\mathcal{S}_1$ to be the class of all monomorphisms in **sSet** and let $\mathcal{S}_2$ be the class of inner fibrations. Lemma A.3.13, Corollary A.2.4 and Lemma A.2.7 imply that the hypothesis of Proposition A.2.6 are satisfied. It follows that the class $\mathcal{M}$ of monomorphisms f such that h is inner anodyne for every inclusion g is saturated. To show that $\mathcal{M}$ contains all the right anodyne maps, it is sufficient to show that $\mathcal{M}$ contains all the morphisms $\Lambda^n_j \subset \Delta^n$ for $0 < j \leq n$.

Fix such an inclusion $f\colon \Lambda^n_j \subset \Delta^n$. Apply Proposition A.2.8 taking $\mathcal{S}_1$ to be

$$\{\emptyset \to \Delta^n, \Lambda^n_j \subset \Delta^n\}$$

Lemma A.2.6 implies that the hypothesis of the proposition are satisfied. It follows that the set $\mathcal{M}'$ of monomorphisms g such that h is inner anodyne whenever f is in $\mathcal{S}_1$ is saturated. To show that $\mathcal{M}'$ contains all the inclusions it is sufficient to show that $\mathcal{M}'$ contains all the inclusions $g\colon \partial\Delta^m \subset \Delta^m$. Now, considering f and g, $h(f,g)$ becomes the inclusion $\Lambda^{n+m+1}_j \subset \Delta^{n+m+1}$ which is inner anodyne since $0 < j \leq n < n+m+1$. Taking $\emptyset \to \Delta^n$ and g, h becomes the inclusion

$$\Lambda^{n+m+1}_{m+j} \simeq \Delta^m \star \Lambda^n_j \subset \Delta^m \star \Delta^n \simeq \Delta^{n+m+1}$$

which is inner anodyne. □

Proposition 2.1.24. *Consider a diagram of simplicial sets*

$$K_0 \subset K \xrightarrow{p} X \xrightarrow{q} S$$

with q an inner fibration; set $r = q \circ p$, $p_0 = p|_{K_0}$, $r_0 = r|_{K_0}$. The induced map

$$X_{p/} \to X_{p_0/} \times_{S_{r_0/}} S_{r/}$$

is a left fibration. If moreover the map q is a left fibration then the induced map

$$X_{/p} \to X_{/p_0} \times_{S_{/r_0}} S_{/r}$$

is a left fibration as well.

Proof. The universal property of overcategories implies that the lifting problem

$$\begin{tikzcd}
\Lambda^n_j \arrow[r] \arrow[d] & X_{p/} \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dotted] & X_{p_0/} \times_{S_{r_0/}} S_{r/}
\end{tikzcd}$$

$(0 \leq j < n)$ is equivalent to the following lifting problem:

$$\begin{tikzcd}
K_0 \star \Lambda^n_j \arrow[rr] \arrow[dd] \arrow[dr] & & K_0 \star \Delta^n \arrow[dl] \arrow[dd] \\
& X \arrow[dd, "q"] & \\
K \star \Lambda^n_j \arrow[rr] \arrow[ur] \arrow[dr] & & K \star \Delta^n \arrow[ul, dotted] \arrow[dl] \\
& S &
\end{tikzcd}$$

which is in turn equivalent to

$$\begin{tikzcd}
(K \star \Lambda^n_j) \coprod (K_0 \star \Delta^n) \arrow[r] \arrow[d] & X \arrow[d, "q"] \\
K \star \Delta^n \arrow[r] \arrow[ur, dotted] & S
\end{tikzcd}$$

Now, the previous lemma implies that

$$(K \star \Lambda^n_j) \coprod_{K_0 \star \Lambda^n_j} (K_0 \star \Delta^n) \to K \star \Delta^n$$

is inner anodyne, so that the last lifting problem has solution. The proof of the other statement is similar. □

Corollary 2.1.25. *Let S be an ∞-category and let $p \colon K \to S$ be any map. Then the natural projection $S_{p/} \to S$ is a left fibration. In particular, $S_{p/}$ is itself an ∞-category.*

We can finally give the following definition:

Definition 2.1.26. Let $p \colon K \to S$ be a morphism of simplicial sets, where S is a quasicategory. The quasicategory $S_{/p}$ of Lemma 2.1.21 is called the overcategory of S with respect to p.

2.1.4 Mapping spaces

We introduced in Definition 2.1.4 the notion of object and edge of an ∞-category S. In the discussion following that definition, we introduced the notion of arrow and equivalence as well. We undertake now the task of organizing the set of arrows from an object to another one into a *space*. We stress the word "space" because it is what we should expect starting with our intuition in higher category theory: morphisms should be parametrized by a space, and homotopies in this space should correspond to higher morphisms in our higher category.

Definition 2.1.27. Let S be a quasicategory and let $X, Y \in S_0$ be objects in S. We define $\mathrm{Map}_S(X, Y)$ to be the simplicial set

$$\mathrm{Hom}^R_S(X, Y) := \mathrm{Hom}_{X,Y}(\Delta^\bullet \star \Delta^0, S)$$

where the subscript means that we are considering only those maps of simplicial sets

$$f\colon \Delta^n \star \Delta^0 \to S$$

such that $f|_{\Delta^n} \equiv X$ and $f|_{\Delta^0} \equiv Y$.

To prove that this gives back a space (i.e. a Kan complex) we need to work out an interesting characterization of Kan complexes. We begin with the following elementary propositions, studying the relationship between left fibrations and equivalences:

Proposition 2.1.28. *Let $p\colon \mathcal{C} \to \mathcal{D}$ be a left fibration of ∞-categories and let $f\colon X \to Y$ be a morphism in $\mathcal{C}$ such that $p(f)$ is an equivalence in $\mathcal{D}$. Then f is an equivalence in $\mathcal{C}$.*

Proof. Let $\overline{g}$ be a homotopy inverse for $p(f)$ and let

$$\begin{tikzcd} & p(Y) \arrow[dr, "\overline{g}"] & \\ p(X) \arrow[ur, "p(f)"] \arrow[rr, "\mathrm{id}_{p(X)}"'] & & p(X) \end{tikzcd}$$

be a 2-simplex in $\mathcal{D}$ attesting that $\overline{g}$ is a right homotopy inverse for $p(f)$. Consider the map $\Lambda^2_0 \to \mathcal{C}$ given by the diagram

$$\begin{tikzcd} & Y \\ X \arrow[ur, "f"] \arrow[r, "\mathrm{id}_X"'] & X \end{tikzcd}$$

Since $p\colon \mathcal{C} \to \mathcal{D}$ is a left fibration, it follows that there exists a right homotopy inverse $g\colon Y \to X$ for f such that $p(g) = \overline{g}$. Since this latter map is a homotopy equivalence, we deduce, via the same argument, that g has a right homotopy inverse. It follows that f is a right homotopy inverse for g, i.e. f is an equivalence. $\square$

Corollary 2.1.29. *Let $p\colon \mathcal{C} \to \mathcal{D}$ be a left fibration of ∞-categories, let Y be an object of $\mathcal{C}$ and let $\overline{f}\colon \overline{X} \to p(Y)$ be an equivalence in $\mathcal{D}$. There exists an equivalence $f\colon X \to Y$ in $\mathcal{C}$ such that $p(f) = \overline{f}$.*

Proof. Let $\overline{g}\colon p(Y) \to \overline{X}$ be a homotopy inverse for $\overline{f}$ and choose $g\colon X \to Y$ in $\mathcal{C}$ such that $p(g) = \overline{g}$ (using the right lifting property of p with respect to $\Lambda^1_0 \subset \Delta^1$. Now consider a 2-simplex in $\mathcal{D}$

$$\begin{tikzcd} & \overline{X} \arrow[dr, "\overline{f}"] & \\ p(Y) \arrow[ur, "p(g)"] \arrow[rr, "\mathrm{id}_{p(Y)}"'] & & p(Y) \end{tikzcd}$$

attesting that $\overline{g} = p(g)$ is a homotopy inverse for $\overline{f}$ and apply the lifting property of p to the map $\Lambda^2_0 \to \mathcal{C}$ given by

$$\begin{tikzcd} & X \\ Y \arrow[ur, "g"] \arrow[r, "\mathrm{id}_Y"'] & Y \end{tikzcd}$$

The lifted map $f\colon X \to Y$ is an equivalence thanks to Proposition 2.1.28. $\square$

Proposition 2.1.30. *Let S be an ∞-category and let $\phi\colon \Delta^1 \to S$ be an edge. ϕ is an equivalence if and only if for every map $f_0\colon \Lambda^n_0 \to S$ such that $f|_{\{0,1\}} = \phi$ there is an extension of f_0 to Δ^n.*

Proof. First of all assume that ϕ is an equivalence. $\square$

Corollary 2.1.31. *Let S be a simplicial set. The following statement are equivalent:*

1. *S is an ∞-category and $\mathrm{h}S$ is a groupoid;*
2. *S has the lifting property with respect to all the horn inclusions $\Lambda^n_i \subset \Delta^n$ for $0 \le i < n$;*
3. *S has the lifting property with respect to all the horn inclusions $\lambda^n_i \subset \Delta^n$ for $0 < i \le n$;*
4. *S is a Kan complex.*

Proof. Lemma 2.1.30 readily implies the equivalence between 1. and 2. A dual argument shows the equivalence between 1. and 3.; now the thesis follows because 4. is equivalent to 2. and 3. $\square$

Proposition 2.1.32. *Let S be a quasicategory. Then $\mathrm{Hom}^R_S(X,Y)$ is a Kan complex.*

Proof. Consider a diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\varphi"] \arrow[d, tail] & \mathrm{Hom}^R_S(X,Y) \\ \Delta^n \arrow[ur, dotted, "\psi"'] & \end{tikzcd}$$

where $0 < i \le n$. Unfolding the definition of $\mathrm{Hom}^R_S(X,Y)$ we see that this problem is equivalent to finding a lift in the following diagram:

$$\begin{tikzcd} \Lambda^{n+1}_i \arrow[r, "\sim"] \arrow[d, tail] & \Lambda^n_i \star \Delta^0 \arrow[r] \arrow[d, tail] & S \\ \Delta^{n+1} \arrow[r, "\sim"] & \Delta^n \star \Delta^0 \arrow[ur, dotted] & \end{tikzcd}$$

so that the previous problem becomes equivalent to

$$\begin{tikzcd} \Lambda^{n+1}_i \arrow[r] \arrow[d, tail] & S \\ \Delta^{n+1} \arrow[ur, dotted] & \end{tikzcd}$$

Since $0 < i < n+1$, we see that the lifting exists by hypothesis. Corollary 2.1.31 shows that $\mathrm{Hom}^R_S(X,Y)$ is a Kan complex. $\square$

Remark 2.1.33. In the notation Hom^R_S the letter R stands for "right". This is because other choices are possible. For example one could introduce, symmetrically:

$$\mathrm{Hom}^L_S(X,Y) := \mathrm{Hom}_{X,Y}(\Delta^0 \star \Delta^\bullet, S)$$

There is no particular reason to choose one instead of the other, just a matter of convenience. However, it is important to remark that the two definitions lead to weakly equivalent simplicial sets. This is not completely obvious, especially because there isn't a obvious comparison map between $\mathrm{Hom}^R_S(X,Y)$ and $\mathrm{Hom}^L_S(X,Y)$, at least to the best of my knowledge. Here we sketch briefly how to do prove the result: we can introduce a third definition, which will appear surely more natural to the reader accustomed with the theory of simplicial sets:

$$\mathrm{Hom}_S(X,Y) := \mathbf{hom}_{X,Y}(\Delta^1, S)$$

where $\mathbf{hom}_{X,Y}$ denotes the subcomplex of $\mathbf{hom}(\Delta^1, S)$ defined by the condition that all its element

$$f \in (\mathbf{hom}(\Delta^1, S))_n = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^1 \times \Delta^n, S)$$

satisfy

$$f|_{\{0\}\times\mathbf{n}} \equiv X, \qquad f|_{\{1\}\times\mathbf{n}} \equiv Y$$

Now, we have natural inclusions of bisimplicial sets

$$\Delta^\bullet \star \Delta^0 \to \Delta^\bullet \times \Delta^1 \leftarrow \Delta^0 \star \Delta^\bullet$$

inducing maps

$$\mathrm{Hom}^R_S(X,Y) \leftarrow \mathrm{Hom}_S(X,Y) \to \mathrm{Hom}^L_S(X,Y)$$

It can be shown that these maps are weak equivalences of simplicial sets.

2.1.5 Interlude II: simplicial nerve

In this second technical interlude we develop the machinery of the simplicial nerve. We will need this construction many times in this mémoire. The most immediate one will be the definition of the notion of *categorical equivalence*, which is the correct notion of equivalence between ∞-categories. In fact, the simplicial nerve will provide a different way to build the homotopy category of a quasicategory S, in such a way that the enrichment over **sSet** will be immediate.

We begin with the following definition:

Definition 2.1.34. Let $(J,<)$ be a finite, nonempty linearly ordered set. If $i,j \in J$ define $P_{i,j}$ to be the partially ordered set

$$P^J_{i,j} = P_{i,j} := \{I \subseteq J \mid i,j \in I \text{ and } \forall k \in I\ i \leq k \leq j\}$$

Fix $(J,<) = (J,<_J)$ as in the above definition. If $i_0 < i_1 < \ldots < i_n$ are elements in J then we have a map

$$\alpha^J_{i_0,\ldots,i_n} \colon P_{i_0,i_1} \times \ldots \times P_{i_{n-1},i_n} \to P_{i_0,i_n}$$

defined by

$$\alpha_{i_0,\ldots,i_n}(I_1,\ldots,I_n) := I_1 \cup \ldots \cup I_n$$

This map is natural in J in the sense that if $(J',<_{J'})$ is another finite nonempty linearly ordered set and $f\colon J \to J'$ is an increasing function, then each choice of elements

$$i_0 <_J i_1 <_J \ldots <_J i_n$$

determines a commutative diagram

$$\begin{tikzcd}
P^J_{i_0,i_1} \times \ldots \times P^J_{i_{n-1},i_n} \arrow[r, "\alpha^J_{i_0,\ldots,i_n}"] \arrow[d, "f"'] & P_{i_0,i_n} \arrow[d, "f"] \\
P^{J'}_{f(i_0),f(i_1)} \times \ldots \times P^{J'}_{i_{n-1},i_n} \arrow[r, "\alpha^{J'}_{f(i_0),\ldots,f(i_n)}"] & P_{f(i_0),f(i_n)}
\end{tikzcd}$$

Consider the following definition:

Definition 2.1.35. Let $(J,<)$ be a finite nonempty linearly ordered set. The simplicial category $\mathfrak{C}[\Delta^J]$ is defined as follows:

1. the objects of $\mathfrak{C}[\Delta^J]$ are the elements of J;

2. if $i, j \in J$ then

$$\mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i,j) := \begin{cases} \emptyset & \text{if } j < i \\ N(P_{i,j}) & \text{if } i \leq j \end{cases}$$

3. if $i_0 \leq i_1 \ldots \leq i_n$ then the composition morphism

$$\mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i_0,i_1) \times \ldots \times \mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i_{n-1},i_n) \to \mathrm{Map}_{\mathfrak{C}[\Delta^J]}(i_0,i_n)$$

is given by $N(\alpha^J_{i_0,\ldots,i_n})$ (observing that N commutes with products).

The assignment $J \mapsto \mathfrak{C}[\Delta^J]$ is functorial in J. It follows that we obtain a functor

$$\mathfrak{C} \colon \mathbf{\Delta} \to \mathbf{Cat}_\Delta$$

where $\mathbf{Cat}_\Delta$ is the category of (small) simplicial categories. In particular, we obtain a cosimplicial object $\mathfrak{C}[\Delta^\bullet]$ in $\mathbf{Cat}_\Delta$; this object can be used in order to define the simplicial nerve:

Definition 2.1.36. Let $\mathcal{C}$ be a simplicial category. We define the simplicial nerve $N(\mathcal{C})$ to be the simplicial set

$$\mathrm{Hom}_{\mathbf{Cat}_\Delta}(\mathfrak{C}[\Delta^\bullet], \mathcal{C})$$

Remark 2.1.37. It is possible to extend the functor $\mathfrak{C}$ to a colimit-preserving functor

$$\mathfrak{C} \colon \mathbf{sSet} \to \mathbf{Cat}_\Delta$$

In fact, the left Kan extension of $\mathfrak{C}$ along the Yoneda embedding $\mathbf{\Delta} \to \mathbf{sSet}$ exists because $\mathbf{Cat}_\Delta$ is cocomplete.

Remark 2.1.38. If S is a simplicial set, $\mathfrak{C}[S]$ is a simplicial category; it can be shown that forgetting the enrichment over **sSet**, $\mathfrak{C}[S]$ is just another model for $\mathfrak{h}S$. This allows to give another definition of the mapping spaces for an ∞-category; the main drawback is that these new mappings spaces need not to be fibrant in general.

The main result concerning the simplicial nerve is the following:

Theorem 2.1.39. *Let $\mathcal{C}$ be a simplicial category such that for every pair of objects $X, Y \in$ $\mathrm{Ob}(\mathcal{C})$ the mapping space $\mathrm{Hom}_\mathcal{C}(X,Y;\mathbf{sSet})$ is a Kan complex. Then $N(\mathcal{C})$ is an ∞-category.*

Proof. Using the adjunction $\mathfrak{C} \dashv N$, we are reduced to show that for every $0 < i < n$, every extension problem

$$\begin{tikzcd} \mathfrak{C}[\Lambda^n_i] \arrow[r] \arrow[d] & \mathcal{C} \\ \mathfrak{C}[\Delta^n] \arrow[ur, dotted] & \end{tikzcd}$$

has a solution. We can review $\mathfrak{C}[\Lambda^n_i]$ as a simplicial subcategory of $\mathfrak{C}[\Delta^n]$; this subcategory contains all the objects of $\mathfrak{C}[\Delta^n]$ and moreover $\mathrm{Map}_{\mathfrak{C}[\Lambda^n_i]}(j,k)$ coincides with $\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(j,k)$ whenever $j \neq 0$ and $k \neq n$. It is therefore sufficient to show that the lifting problem

$$\begin{tikzcd} \mathrm{Map}_{\mathfrak{C}[\Lambda^n_i]}(0,n) \arrow[r] \arrow[d] & \mathrm{Hom}_\mathcal{C}(X,Y;\mathbf{sSet}) \\ \mathrm{Map}_{\mathfrak{C}[\Delta^n]}(0,n) \arrow[ur, dotted] & \end{tikzcd}$$

has solution. Since $\mathrm{Hom}_\mathcal{C}(X,Y;\mathbf{sSet})$ is a Kan complex, we are reduced to show that the map on the left is a standard trivial cofibration. This follows by a direct verification: $\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(0,n)$ can be identified with a cube $(\Delta^1)^{n-1}$, and under this identification $\mathrm{Map}_{\mathfrak{C}[\Lambda^n_i]}(0,n)$ is obtained by removing the interior and one face. □

We conclude this section giving the following definition:

Definition 2.1.40. A morphism of ∞-categories $F \colon T \to S$ is said to be a *categorical equivalence* if the induced functor $\mathrm{h}\mathfrak{C}[F] \colon \mathrm{h}\mathfrak{C}[T] \to \mathrm{h}\mathfrak{C}[S]$ is an equivalence of simplicial categories.

2.1.6 Functor categories

In higher category theory, the task of giving a correct definition of higher functor is not easy, depending on the approach. The problem is that one should specify a large amount of additional data (besides the obvious functions sending an n-arrow into another n-arrow), whose purpose is to attest the commutativity of certain diagrams in a coherent way. Limiting ourselves to $(\infty, 1)$-categories and working in the framework of quasicategories, this task is not difficult at all. A higher functor will be simply a morphism of simplicial sets. An intuitive explanation will be given later, after the discussion about mapping space.

Assuming that this gives the correct notion of higher functor, another problem is to deal with the collection of all higher functors between two $(\infty, 1)$-categories. As in standard category theory we can form a category out of functors between categories (whose morphisms are given by natural transformations), we aim to do the same thing in the higher categorical context. This gives a first motivation for the following definition:

Definition 2.1.41. Let S be a quasicategory and let K be any simplicial set. We define the space of (higher) functors from K to S to be

$$\mathrm{Fun}(K, S) := \mathbf{hom}(K, S)$$

In order to have a good definition, we should prove the following theorem:

Theorem 2.1.42. *If S is a quasicategory, then* $\mathrm{Fun}(K, S) = \mathbf{hom}(K, S)$ *is a quasicategory for every simplicial set K.*

Proof. The theory of anodyne extensions allows to check that $\mathrm{Fun}(K, S)$ has the lifting property with respect to every inner anodyne extension $A \subset B$. A standard adjunction argument shows that the following two lifting problems are equivalent:

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & \mathrm{Fun}(K,S) \\ B \arrow[ur, dotted] & \end{tikzcd} \qquad \begin{tikzcd} A \times K \arrow[r] \arrow[d, tail] & S \\ B \times K \arrow[ur, dotted] & \end{tikzcd}$$

Corollary 2.1.20 implies now that $A \times K \to B \times K$ is inner anodyne. Since S is an ∞-category, the second problem has solution, and the thesis follows. □

2.1.7 Limits and colimits

The next step in building our framework will be to define the notion of limit and colimit of a ∞-functor. As in classical category theory, the best way to define limits and colimits is to reduce to the case of initial and final objects with the machinery of overcategories and undercategories. In our framework the latter notions have already been defined, so we consider initial and final objects.

Definition 2.1.43. Let S be a quasicategory and let X be a vertex of S. We say that X is final in S if the canonical map $p\colon S_{/X} \to S$ is a trivial Kan fibration.

To explain how this definition generalizes the classical one we need to work out a property of left fibrations. In particular, we need the following:

Lemma 2.1.44. *Let $p\colon S \to T$ be a left fibration of simplicial sets. Suppose that for every vertex $t \in T$ the fiber S_t is contractible. Then p is a trivial Kan fibration.*

Proof. See [HTT, Lemma 2.1.3.4]. □

Remark 2.1.45. In [HTT] this Lemma is a key point in proving an interesting characterization of Kan fibrations. We don't reproduce the proof, but we state the result: a left fibration $f\colon S \to T$ of simplicial sets is a Kan fibration if and only if for every edge $f\colon t \to t'$ in T the induced map $f_!\colon S_t \to S_{t'}$ is an isomorphism in $\mathrm{Ho}(\mathbf{sSet})$.

Proposition 2.1.46. *Let S be a quasicategory and let X be a vertex of S. X is final in S if and only if for every other vertex Y of S,* $\mathrm{Hom}^R_S(Y, X)$ *is a contractible Kan complex.*

Proof. We claim that the fiber of $p\colon S_{/X} \to S$ over Y is precisely $\mathrm{Hom}^R_S(Y,X)$. In fact, this fiber is spanned by those simplexes sent to the constant subcomplex generated by Y. Under the identification

$$\mathrm{Hom}(\Delta^n, S_{/X}) \simeq \mathrm{Hom}_X(\Delta^n \star \Delta^0, S) \simeq \mathrm{Hom}_X(\Delta^{n+1}, S)$$

we see that for $\omega \in \mathrm{Hom}(\Delta^n, S_{/X})$, $p(\omega)$ corresponds to $\omega|_{\{0,\dots,n\}}$. If $\omega \in p^{-1}(Y)$, then $\omega|_{\{0,\dots,n\}} = Y$, i.e. $\omega \in \mathrm{Hom}^R_S(Y,X)$. The other inclusion being trivial, we completed the proof of our claim.

Now, if p is a trivial fibration, $\mathrm{Hom}^R_S(Y,X)$ is a contractible Kan complex. For the converse, observe first of all that Proposition 2.1.24 implies that

$$p\colon S_{/X} \to S$$

is a left fibration. Since its fibers are contractible Kan complexes by hypothesis, Lemma 2.1.44 implies that p is a trivial Kan fibration. □

Corollary 2.1.47. *Let S be a quasicategory and let X be a final object in S. For every other object Y there is a 1-simplex $f\colon \Delta^1 \to S$ (unique up to a contractible space of choices) such that $d_0 f = X$ and $d_1 f = Y$.*

Proof. A contractible Kan complex is in particular nonempty (it has the lifting property with respect to $\emptyset = \partial\Delta^0 \to \Delta^0$, hence the statement is a consequence of the previous proposition. □

Definition 2.1.48. Let S be an ∞-category and let $p\colon K \to S$ be a morphism of simplicial sets. A limit for p is a final object in the category $S_{/p}$. Dually, a colimit for p is an initial object in the category $S_{p/}$.

We will show at the end of next section that this notion of (co)limit has a homotopical significance. Namely, we will show that to every model category we can associate an ∞-category and that under this correspondence, (co)limits corresponds exactly to homotopy (co)limits.

We conclude giving the analogue in the ∞-categorical setting of a some well known results of classical category theory:

Proposition 2.1.49. *Let S be an ∞-category and suppose we are given a map $\sigma\colon \Delta^2 \times \Delta^1 \to \mathcal{C}$, depicted as a dagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z' \end{tikzcd}$$

Suppose that the left square is a pushout in S. Then the right square is a pushout if and only if the outer square is a pushout.

Notation 2.1.50. If K is a simplicial set we will denote by $K^{\triangleright}$ the simplicial set $K \star \Delta^0$. Similarly, we will denote by $K^{\triangleleft}$ the simplicial set $\Delta^0 \star K$.

Proposition 2.1.51. *Let K and S be simplicial sets and let $\mathcal{C}$ be an ∞-category which admits k-indexed colimits. Then:*

1. *the ∞-category $\mathrm{Fun}(S,\mathcal{C})$ admits K-indexed colimits;*
2. *a map $K^{\triangleright} \to \mathrm{Fun}(S,\mathcal{C})$ is a colimit diagram if and only if for each vertex $s \in S$, the induced map $K^{\triangleright} \to \mathcal{C}$ is a colimit diagram.*

Proof. See [HTT, Corollary 5.1.2.3]. □

2.1.8 Yoneda lemma

Notation 2.1.52. Consider the model category **sSet**, with the standard model structure. This category is a simplicial category; let Kan be the full subcategory of fibrant-cofibrant objects in Kan; this is still a simplicial category and the hypothesis of Theorem 2.1.39 are satisfied, so that the nerve $N(\mathrm{Kan})$ is an ∞-category. We will denote it by $\mathcal{S}$ and we will refer to that as the ∞-category of spaces.

Given any ∞-category X, we can therefore consider the category of (simplicial) presheaves on X:

$$\mathcal{P}(X) := \mathrm{Fun}(X^{\mathrm{op}}, \mathcal{S})$$

We would like to construct an embedding $j\colon X \to \mathcal{P}(X)$. One possibility is the following: assume that $\mathcal{C}$ is a simplicial category. Then we can construct a functor toward the category of Kan complexes:

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Kan}$$

given by the formula

$$(X, Y) \mapsto \mathrm{Sing}|\mathrm{Hom}_{\mathcal{C}}(X, Y; \mathbf{sSet})|$$

If we take $\mathcal{C} = \mathfrak{C}[X]$, using the universal property of the product to obtain a functor

$$\mathfrak{C}[X^{\mathrm{op}} \times X] \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Kan}$$

Passing to the simplicial nerve we get a functor

$$X^{\mathrm{op}} \times X \to \mathcal{S} = N(\mathrm{Kan})$$

which can be seen as a functor

$$j\colon X \to \mathrm{Fun}(X^{\mathrm{op}}, \mathcal{S})$$

Definition 2.1.53. Let X be an ∞-category. The Yoneda embedding of X is the functor $j\colon X \to \mathrm{Fun}(X^{\mathrm{op}}, \mathcal{S})$ built above.

In order to check that this construction is a good generalization of the standard categorical construction, one has to be sure that the following result holds:

Proposition 2.1.54. *Let X be an ∞-category. The Yoneda embedding $j\colon X \to \mathcal{P}(X)$ is fully faithful.*

Proof. See [HTT, Proposition 5.1.3.1]. □

Next, one can also prove that the Yoneda embedding j preserves small limits:

Proposition 2.1.55. *Let X be a small ∞-category and let $j\colon X \to \mathcal{P}(X)$ be the Yoneda embedding. Then j preserves all small limits which exist in X.*

Proof. See [HTT, Proposition 5.1.3.2]. □

In particular, combining (the dual of) this result with Proposition 2.1.51 we obtain that for every object $x \in \mathrm{Ob}(X)$, the corresponding functor $j(x)$ commutes with colimits. Observe that this functor is *informally* expressed by the formula

$$y \mapsto \mathrm{Map}_X(y, x)$$

We end this section by sketching the construction of another model for $\mathcal{P}(X)$, where X is an ∞-category. Let $\mathcal{P}'(X)$ be the nerve of the full subcategory of $\mathbf{sSet}_{/X}$ spanned by the right fibrations $Y \to X$. The important result is that the categories $\mathcal{P}(X)$ and $\mathcal{P}'(X)$ are equivalent; however, the proof, which essentially consists in the generalization of the Grothendieck construction in the ∞-categorical setting, uses tools we haven't developed (more precisely, the straightening and unstraightening functors); we therefore state it as a result:

Proposition 2.1.56. *If X is an ∞-category there is an equivalence of ∞-categories between $\mathcal{P}(X)$ and $\mathcal{P}'(X)$.*

Proof. See [HTT, Proposition 5.1.1.1]. □

2.1.9 Interlude III: Cartesian fibrations

The goal of this technical interlude is to develop the machinery necessary to the introduction of the notion of adjoint morphisms. The goal is to introduce the definition of cartesian fibration, and to work out some of its basic properties. In Section 2.1.2 we saw that right fibrations generalize the notion of category cofibered in groupoids in our ∞-categorical setting. Here we undertake the problem of giving a satisfying generalization of (co)fibered category.

Cartesian morphisms

The first thing to do is to provide a notion of (co)cartesian morphism. Before diving into the definition, let us make a simple observation: one of the main features of fibered categories is their equivalence with pseudo–functors. To get something similar, we should consider only morphisms of ∞-categories whose fibers are again ∞-categories. For this reason, we shall restrain ourselves to inner fibrations.

Definition 2.1.57. Let $p\colon X \to S$ be an inner fibration of simplicial sets. An arrow $f\colon x \to y$ in X is said to be *p–Cartesian* if the induced map

$$X_{/f} \to X_{/y} \times_{S_{/p(y)}} S_{/p(f)} \tag{2.4}$$

is a trivial Kan fibration.

Lemma 2.1.58. *Let $p\colon X \to S$ be an inner fibration of simplicial sets. An edge f in X is p–Cartesian if and only if the dotted arrow in the diagram*

$$\begin{tikzcd} \Delta^1 \arrow[d, "{[n+1,n+2]}"'] \arrow[dr, "f"] & \\ \Lambda^{n+2}_{n+2} \arrow[r] \arrow[d, tail] & X \arrow[d, "p"] \\ \Delta^{n+2} \arrow[r] \arrow[ur, dotted] & S \end{tikzcd} \tag{2.5}$$

exists for every $n \geq 0$.

Proof. We have to show that for every $n \geq 0$ the lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X_{/f} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & X_{/y} \times_{S/p(y)} S_{/p(f)} \end{tikzcd} \tag{2.6}$$

is equivalent to the lifting problem (2.5). The universal property of overcategories implies immediately that the lifting problem 2.6 is equivalent to the following pair of diagrams

$$\begin{tikzcd} \Lambda^{n+1}_{n+1} \arrow[rr] \arrow[dr] \arrow[dd, "\alpha"'] & & \Delta^{n+1} \arrow[dl] \arrow[dd] \\ & X \arrow[dd, "p"] & \\ \partial\Delta^n \star \Delta^1 \arrow[ur] \arrow[rr] \arrow[dr] & & \Delta^{n+2} \arrow[ul, dotted] \arrow[dl] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} \Delta^1 \arrow[d] \arrow[dr, "f"] & \\ \partial\Delta^n \star \Delta^1 \arrow[r] & X \end{tikzcd} \tag{2.7}$$

where the map α is defined as (cfr. Lemma A.3.13.4):

$$\Lambda^{n+1}_{n+1} \simeq \partial\Delta^n \star \Delta^0 \xrightarrow{\mathrm{id}\star d^0} \partial\Delta^n \star \Delta^1$$

Using the fact that the geometric realization reflects colimits (cfr. Corollary A.1.3) we obtain that the following diagram is a pushout in **sSet**:

$$\begin{tikzcd} \Lambda^{n+1}_{n+1} \arrow[r] \arrow[d, "\alpha"'] & \Delta^{n+1} \arrow[d] \\ \partial\Delta^n \star \Delta^1 \arrow[r] & \Lambda^{n+2}_{n+2} \end{tikzcd}$$

and that the induced map $\Delta^1 \to \partial\Delta^n \star \Delta^1 \to \Lambda^{n+2}_{n+2}$ is exactly the map

$$[n+1, n+2] \colon \Delta^1 \to \Lambda^{n+2}_{n+2}$$

It follows immediately that the lifting problem (2.7) is equivalent to the lifting problem (2.5), so that the lemma follows. □

This lemma allows to show that the familiar properties of cartesian morphisms are satisfied in the ∞-categorical setting:

Corollary 2.1.59. *1. If $p \colon X \to S$ is an isomorphism of simplicial sets, every edge of X is p-Cartesian;*

2. let

$$\begin{tikzcd} X' \arrow[r, "q"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ S' \arrow[r] & S \end{tikzcd}$$

be a pullback diagram of simplicial sets, where p is an inner fibration. If f is an edge of X' such that $q(f)$ is p-Cartesian, then f is p'-Cartesian.

3. let $p \colon X \to Y$ and $q \colon Y \to Z$ be inner fibrations and let $f \colon x' \to x$ be an edge of X such that $p(f)$ is q-Cartesian. Then f is p-Cartesian if and only if f is $(q \circ p)$-Cartesian.

Proof. *1.* and *2.* are straightforward consequences of Lemma 2.1.58. The proof of *3.* is not hard, but we refer to [HTT, Proposition 2.4.1.3] since we won't need it in this mémoire. □

Cartesian fibrations

Definition 2.1.60. A map $p \colon X \to S$ of simplicial sets is said to be a *Cartesian fibration* if the following conditions are satisfied:

1. the map p is an inner fibration;

2. for every edge $f \colon x \to y$ of S and every vertex $\widetilde{y}$ of X with $p(\widetilde{y}) = y$, there is a p-Cartesian edge $\widetilde{f} \colon \widetilde{x} \to \widetilde{y}$ such that $p(\widetilde{f}) = f$.

The main result concerning cartesian fibrations consists in showing that given a Cartesian fibration $p \colon X \to S$ there is a functor going from S to an ∞-category of ∞-categories. This is an analogue of the familiar equivalence between fibered categories over a category S and functors from S to **Cat**. However, the proof of this result use a quite refined technique (marked fibrations of simplicial sets), and so we will omit the proof.

2.1.10 Adjunctions

Definition 2.1.61. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. An *adjunction* between $\mathcal{C}$ and $\mathcal{D}$ is a map $q\colon \mathcal{M} \to \Delta^1$ which is both a Cartesian fibration and a coCartesian fibration together with equivalences $\mathcal{C} \to \mathcal{M}_{\{0\}}$ and $\mathcal{D} \to \mathcal{M}_{\{1\}}$.

In order to be able to switch easily from cartesian fibrations to functors, one should develop the theory of straightening and unstraightening functors, which is beyond the scope of this mémoire. We will prove the existence of an adjunction later on using an *ad hoc* construction, but we strongly recommend to look at the treatment of this subject given in [HTT, Chapter 5.2].

Still, we will need the existence of a unit transformation attached to every adjunction. Let us therefore give the following definition:

Definition 2.1.62. Suppose we are given a pair of functors

$$\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{D}$$

between ∞-categories. A *unit transformation* for (f, g) is a morphism $u\colon \mathrm{id}_{\mathcal{C}} \to g \circ f$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ with the following property: for every pair of objects $C \in \mathrm{Ob}(\mathcal{C})$, $D \in \mathrm{Ob}(\mathcal{D})$, the composition

$$\mathrm{Map}_{\mathcal{D}}(f(C), D)\mathrm{Map}_{\mathcal{C}}(g(f(C)), g(D) \xrightarrow{u(C)} \mathrm{Map}_{\mathcal{C}}(C, g(D))$$

is an isomorphism in the homotopy category of spaces $\mathrm{Ho}(\mathbf{sSet})$.

The main result concerning unit transformation is the following:

Theorem 2.1.63. *Let $f\colon \mathcal{C} \to \mathcal{D}$ and $g\colon \mathcal{D} \to \mathcal{C}$ be a pair of functors between ∞-categories $\mathcal{C}$ and $\mathcal{D}$. The following conditions are equivalent:*

1. *the functor f is a left adjoint to g;*

2. *there exists a unit transforamtion $u\colon \mathrm{id}_{\mathcal{C}} \to g \circ f$.*

Proof. See [HTT, Proposition 5.2.2.8]. □

Corollary 2.1.64. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories and let $f\colon \mathcal{C} \to \mathcal{D}$ and $g\colon \mathcal{D} \to \mathcal{C}$ be adjoint functors. Then f and g induce adjoint functors $\mathrm{h}f\colon \mathrm{h}\mathcal{C} \rightleftarrows \mathrm{h}\mathcal{D}\colon \mathrm{h}g$ between $\mathrm{Ho}(\mathbf{sSet})$-enriched categories.*

Proof. This is a clear consequence of Theorem 2.1.63, since the unit transformation $u\colon \mathrm{id}_{\mathcal{C}} \to g \circ f$ induces a unit transformation $\mathrm{h}u\colon \mathrm{id}_{\mathrm{h}\mathcal{C}} \to (\mathrm{h}g) \circ (\mathrm{h}f)$. □

Corollary 2.1.65. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be right adjoint to a functor $g\colon \mathcal{D} \to \mathcal{C}$. Then f is a categorical equivalence if and only if*

1. *f reflects equivalences;*

2. *the unit transformation $u\colon \mathrm{Id}_{\mathcal{D}} \to g \circ f$ is an equivalence.*

Proof. This is a consequence of previous corollary. In fact, if f reflects equivalences, then $\mathrm{h}f$ reflects isomorphisms, and if the unit transformation is an equivalence, then the induced unit of $\mathrm{h}g \dashv \mathrm{h}f$ is an isomorphism. Since f is a categorical equivalence if and only if $\mathrm{h}f$ is an equivalence of enriched categories, the result follows from the analogous statement in classical enriched category theory (where it is a trivial consequence of the triangular identities). □

2.1.11 Interlude IV: presentable ∞-categories

In this last technical interlude, we develop the machinery of presentable ∞-categories. Throughout this mémoire we will make really often this assumption on the ∞-categories we are using; the idea behind such technical property is that the category is large enough to have small limits and colimits, but at the same time we can completely control it via a small subcategory, making results as the adjoint functor theorem true in the context of presentable ∞-categories. In order to give the definition and develop the first basic properties, we will need to review the notion of filtered simplicial set and of ind-completion; after that, we will introduce the notion of accessible ∞-category and finally the one of presentable ∞-category.

Ind completion

First of all, we will need to translate in the language of ∞-categories the definition of filtered diagram. This generalization is quite easy to obtain:

Definition 2.1.66. Let κ be a regular cardinal and let $\mathcal{C}$ be an ∞-category. We will say taht $\mathcal{C}$ is κ-filtered if, for every κ-small simplicial set K and every map $f\colon K \to \mathcal{C}$ there exists a map $\overline{f}\colon K^{\triangleright} \to \mathcal{C}$ extending f.

Suppose now we are given an ∞-category $\mathcal{C}$ and a regular cardinal κ. We would like to define a new category $\widetilde{\mathcal{C}}$ containing $\mathcal{C}$ as a full subcategory and such that it is complete under κ-filtered colimits. We can use, as in ordinary category theory, the Yoneda embedding $j\colon \mathcal{C} \to \mathcal{P}(\mathcal{C})$; the best way to define the completion we are looking for is to employ the equivalence of Proposition 2.1.56: we will define $\mathrm{Ind}_\kappa(\mathcal{C})$ to be the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ spanned by those morphisms classifying right fibrations $\mathcal{D} \to \mathcal{C}$, where $\mathcal{D}$ is κ-filtered.

Proposition 2.1.67. *Let $\mathcal{C}$ be a small ∞-category and let κ be a regular cardinal. The full subcategory $\mathrm{Ind}_\kappa(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})$ is stable under κ-filtered colimits.*

Proof. See [HTT, Proposition 5.3.5.3]. □

Accessible and presentable ∞-categories

An accessible ∞-category is an ∞-category that can be obtained as the ind completion of some small ∞-category.

Definition 2.1.68. Let κ be a regular cardinal. An ∞-category $\mathcal{C}$ is *κ-accessible* if there exists a small ∞-category $\mathcal{C}^0$ and an equivalence

$$\mathrm{Ind}_\kappa(\mathcal{C}^0) \to \mathcal{C}$$

We will say that $\mathcal{C}$ is *accessible* if it is κ-accessible for some regular cardinal κ.

Definition 2.1.69. If $\mathcal{C}$ is an accessible ∞-category, then a functor $F\colon \mathcal{C} \to \mathcal{C}'$ is *accessible* if it preserves κ-filtered colimits for some regular cardinal κ.

Definition 2.1.70. An ∞-category $\mathcal{C}$ is said to be *presentable* if it is accessible and moreover it admits small colimits.

The presentability hypothesis allows to proof deep theorems in the theory of ∞-categories, such as the analogue of the adjoint functor theorem:

Theorem 2.1.71. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories.*

1. *The functor F has a right adjoint if and only if it preserves small colimits;*

2. *the functor F has a left adjoint if and only if it is accessible and preserves small limits.*

Proof. See [HTT, Corollary 5.5.2.9]. □

2.1.12 The small object argument

In this last subsection about the language of quasicategories, we will present an ∞-categorical generalization of the well-known *small object argument*. This result will play a major role in the proof of the main theorem of Chapter 4, since it will allow to begin a reduction argument.

Theorem 2.1.72. *Let $\mathcal{C}$ be a presentable ∞-category and let S be a small collection of morphisms in $\mathcal{C}$. Then every morphism $f\colon X \to Z$ admits a factorization*

$$X \xrightarrow{f'} Y \xrightarrow{f''} Z$$

where f' is a transfinite pushout of morphisms in S and f'' has the RLP with respect to S.

Proof. Write $S = \{g_i \colon C_i \to_i\}_{i \in I}$. Choose a regular cardinal κ such that each of the objects C_i is κ-compact. We will construct by transfinite induction a diagram $F\colon N[\kappa] \to \mathcal{C}_{/Z}$, where $[\kappa]$ denotes the linearly ordered set of ordinals $\{\beta \colon \beta \leq \alpha\}$. Set F_0 to be the morphism $f\colon X \to Z$; for a nonzero limit ordinal $\lambda \leq \kappa$, we let F_λ be a colimit of the diagrams obtained from the maps $\{F_\alpha\}_{\alpha<\lambda}$. Assume now that $\alpha < \kappa$ and that F_α has been constructed. Then $F_\alpha(\alpha)$ corresponds to a map $X' \to Z$; let $T(\alpha)$ be a set of representatives for all equivalence classes of diagrams σ_t:

$$\begin{tikzcd} C_t \arrow[r] \arrow[d, "g_t"] & X' \arrow[d] \\ D_t \arrow[r] & Z \end{tikzcd}$$

where g_t is a morphism in S. Choose a pushout diagram

$$\begin{tikzcd} \coprod_{t\in T(\alpha)} C_t \arrow[r] \arrow[d] & X' \arrow[d] \\ \coprod_{t\in T(\alpha)} D_t \arrow[r] & X'' \end{tikzcd}$$

in $\mathcal{C}_{/Z}$. We look at X'' as an object of $(C_{X'/})_{/Z}$. The natural map

$$(\mathcal{C}_{/Z})_{F_\alpha/} \to (\mathcal{C}_{X'/})_{/Z}$$

is obviously a trivial Kan fibration,[1] so that we can lift X'' to an object of $(\mathcal{C}_{/Z})/_{F_\alpha/}$, which determines the desired map $F_{\alpha+1}$.

For each $\alpha \leq \kappa$, let $f_\alpha \colon Y_\alpha \to Z$ be the image $F(\alpha) \in \mathcal{C}_{/Z}$. Let $Y = Y_\kappa$ and $f'' = f_\kappa$. To show that f'' has the RLP with respect to every morphism in S, we can equivalently show that for each $i \in I$ and every map $D_i \to Z$ the induced map

$$\mathrm{Map}_{\mathcal{C}_{/Z}}(D_i, Y) \to \mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y)$$

is surjective on connected components. Choose a point $\eta \in \mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y)$; since C_i is κ-compact, the space $\mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y)$ can be realized as the filtered colimit of mapping spaces

$$\mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y) \simeq \varinjlim_{\alpha} \mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y_\alpha)$$

so that we may assume that η is the image of $\eta_\alpha \in \mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y_\alpha)$ for some $\alpha < \kappa$. This point determines a commutative square

$$\begin{tikzcd} C_i \arrow[r] \arrow[d, "g_i"] & Y_\alpha \arrow[d] \\ D_i \arrow[r] & Z \end{tikzcd}$$

[1]Proposition 2.1.24 implies that it is a left fibration, and the fibres are obviously non-empty and contractible.

which is equivalent to σ_t for some $t \in T(\alpha)$. Therefore the image of η_α in $\mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y_{\alpha+1})$ extends to D_i, showing that η lies in the image of

$$\mathrm{Map}_{\mathcal{C}_{/Z}}(D_i, Y_{\alpha+1}) \to \mathrm{Map}_{\mathcal{C}_{/Z}}(C_i, Y)$$

The morphism $F(0) \to F(\kappa)$ in $\mathcal{C}_{/Z}$ induces a morphism $f'\colon X \to Y$ in $\mathcal{C}$; if we can show that f' is a transfinite pushout of morphisms in S, the proof will be complete. Since f' is the transfinite pushout of the maps $Y_\alpha \to Y_{\alpha+1}$, we are reduced to show that such maps are transfinite pushouts of morphisms in S. Choose a well-ordering of $T(\alpha)$, corresponding to an ordinal β. For $\gamma < \beta$, let t_γ denote the corresponding element of $T(\alpha)$. We define a functor $G\colon N[\beta] \to \mathcal{C}$ in such a way that, for every $\beta' \leq \beta$ we have a pushout diagram

$$\begin{tikzcd} \coprod_{\gamma<\beta'} C_{t_\gamma} \arrow[r] \arrow[d] & Y_\alpha \arrow[d] \\ \coprod_{\gamma<\beta'} D_{t_\gamma} \arrow[r] & G(\beta') \end{tikzcd}$$

This exhibits $Y_\alpha \to Y_{\alpha+1}$ as a transfinite pushout of morphisms in S. □

Corollary 2.1.73. *Let $\mathcal{C}$ be a presentable ∞-category and let S be a small collection of morphisms in $\mathcal{C}$. Let Y be any object of $\mathcal{C}$ and let $\phi\colon \mathcal{C}_{/Y} \to \mathcal{C}$ be the forgetful functor. There exists a simplicial object $X_\bullet$ of $\mathcal{C}_{/Y}$ with the following properties:*

1. *for each $n \geq 0$ let $u_n\colon L_n(X_\bullet) \to X_n$ be the canonical map. Then $\phi(u_n)$ is a transfinite pushout of mosphisms in S;*

2. *for each $n \geq 0$, let $v_n\colon X_n \to M_n(X_\bullet)$ be the canonical map in $\mathcal{C}_{/Y}$. Then $\phi(v_n)$ has the RLP with respect to every morphism in S.*

Proof. We construct by induction on n a compatible family of diagrams $X_\bullet^{(n)}\colon N(\mathbf{\Delta}_{\leq n})^{\mathrm{op}} \to \mathcal{C}_{/Y}$. If $n = -1$, we take the constant diagram at $\emptyset$, the initial object of $\mathcal{C}$. Assume now that $n \geq 0$ and that $X_\bullet^{(n-1)}$ has been constructed. Therefore we have well defined latching objects and matching objects $L_n(X)$, $M_n(X)$ as well as a map

$$t\colon L_n(X) \to M_n(X)$$

To extend in a compatible way the diagram $X_\bullet^{(n-1)}$ is sufficient to produce a factorization

$$\begin{tikzcd} & X_n \arrow[dr, "v_n"] & \\ L_n(X_\bullet) \arrow[ur, "u_n"] \arrow[rr, "t"] & & M_n(X_\bullet) \end{tikzcd}$$

in $\mathcal{C}_{/Y}$. However, the dual version of Corollary 2.1.25 implies that the map $\phi\colon \mathcal{C}_{/Y} \to \mathcal{C}$ is a right fibration, so that this problem becomes equivalent to the problem of producing a commutative diagram

$$\begin{tikzcd} & K_n \arrow[dr] & \\ \phi(L_n(X_\bullet)) \arrow[ur, "u"] \arrow[rr, "\phi(t)"] & & \phi(M_n(X_\bullet)) \end{tikzcd}$$

in the ∞-category $\mathcal{C}$. Theorem 2.1.72 implies that it is possible to produce a similar factorization in such a way that conditions *1.* and *2.* are satisfied. □

2.2 From model categories to quasicategories

It is well known that the framework of model categories developed by Quillen is a really powerful context where to speak about abstract homotopy theory. The idea is that a model category provides a "presentation" of a homotopy theory; the main problem is to understand how to extract these homotopical informations from $\mathcal{M}$ and to avoid forgetting something. For example, it is clear that the construction of the homotopy category forgets too much: essentially, it preserves only 1-homotopical informations.

Mapping spaces suggest that every model category is "almost enriched" over **sSet**; there is a more refined construction, the Dwyer-Kan localization that realizes this enrichment in a proper way. This provides a way of turning a model category into a simplicial model category. At this point, the general machinery of the homotopy-coherent nerve can be employed to convert our data into a quasicategory.

2.2.1 O-categories

Generalities

We will fix a Grothendieck universe $\mathbb{U}$ throughout this section; **Cat** will denote the category of all $\mathbb{U}$-small categories. Let O be a *fixed* $\mathbb{U}$-small set. We will denote by O-**Cat** the subcategory of **Cat** whose objects are categories $\mathcal{C}$ satisfying $\mathrm{Ob}(\mathcal{C}) = O$ and whose morphisms are the functors inducing the identity on objects.

Lemma 2.2.1. *The category O-**Cat** has binary products.*

Proof. Let $\mathcal{C}, \mathcal{D} \in \mathrm{Ob}(O\text{-}\mathbf{Cat})$ be two O-categories. Define $\mathcal{C} \times \mathcal{D}$ as the category whose set of objects is O and whose morphisms are defined by

$$\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{D}}(X,Y)$$

Identities and composition are defined componentwise. It is straightforward to check that $\mathcal{C} \times \mathcal{D}$ is a well defined category. Let us check the universal property of the product: first of all we have obviously defined functors of O-categories

$$p \colon \mathcal{C} \times \mathcal{D} \to \mathcal{C}, \qquad q \colon \mathcal{C} \times \mathcal{D} \to \mathcal{D}$$

If $F \colon \mathcal{A} \to \mathcal{C}$ and $G \colon \mathcal{A} \to \mathcal{D}$ are arbitrary functors of O-categories, we define $F \times G \colon \mathcal{A} \to \mathcal{C} \times \mathcal{D}$ to be the identity on objects and

$$(F \times G)(\alpha, \beta) := (F(\alpha, G(\beta))$$

The functoriality is obvious; it is also clear that $p \circ (F \times G) = F$, $q \circ (F \times G) = G$. Finally the uniqueness of $F \times G$ is immediate. ☐

Lemma 2.2.2. *The category O-**Cat** has coproducts.*

Proof. We show that O-**Cat** has *binary* coproducts. A similar argument shows the existence of arbitrary coproducts. Let $\mathcal{C}, \mathcal{D} \in \mathrm{Ob}(O\text{-}\mathbf{Cat})$ be two O-categories. Define $\mathcal{C} * \mathcal{D}$ in the following way: first of all, $\mathrm{Ob}(\mathcal{C} * \mathcal{D})$ is O; next, set

$$\mathrm{Arr}(\mathcal{C})^* := \mathrm{Arr}(\mathcal{C}) \setminus \mathrm{Ob}(\mathcal{C})$$
$$\mathrm{Arr}(\mathcal{D})^* := \mathrm{Arr}(\mathcal{D}) \setminus \mathrm{Ob}(\mathcal{D})$$

where we think the element $x \in \mathrm{Ob}(\mathcal{C})$ in $\mathrm{Arr}(\mathcal{C})$ as id_x. Consider

$$S := \mathrm{Arr}(\mathcal{C})^* \sqcup \mathrm{Arr}(\mathcal{D})^*$$

and

$$T = \coprod_{n \in \mathbb{N}} S^n$$

We will say that an element $(f_n, \ldots, f_0) \in T$ is reduced if for each $i \in \{0, \ldots, n-1\}$, $f_i \in \mathrm{Arr}(\mathcal{C})$ implies $f_{i+1} \in \mathrm{Arr}(\mathcal{D})$. Let $\overline{T}$ denote the subset of reduced elements of T. Given $x, y \in O$ define

$$\mathrm{Hom}'_{\mathcal{C}*\mathcal{D}}(x,y) := \{(f_n, f_{n-1}, \ldots, f_0) \in \overline{T} \mid f_i \colon z_i \to z_{i+1},\ z_0 = x,\ z_{n+1} = y\}$$

Finally set

$$\mathrm{Hom}_{\mathcal{C}*\mathcal{D}}(x,y) := \begin{cases} \mathrm{Hom}'_{\mathcal{C}*\mathcal{D}}(x,y) & \text{if } x \neq y \\ \mathrm{Hom}'_{\mathcal{C}*\mathcal{D}}(x,y) \cup \{\mathrm{id}_x\} & \text{if } x = y \end{cases}$$

The composition is defined in the obvious way, and it is straightforward to check that $\mathcal{C} * \mathcal{D}$ is a category. Observe that $\mathcal{C} * \mathcal{D}$ is a $\mathbb{U}$-small category because we have the following bound

$$|\mathrm{Hom}'_{\mathcal{C}*\mathcal{D}}(x,y)| \leq \left| \bigcup_{n\in\mathbb{N}} \prod_{(x_i)\in O^n} \mathrm{Hom}_{\mathcal{C}}(x_i, x_{i+1}) \cup \mathrm{Hom}_{\mathcal{D}}(x_i, x_{i+1}) \right|$$

and since O is $\mathbb{U}$-small, the set on the right is still $\mathbb{U}$-small.

Let us check that $\mathcal{C} * \mathcal{D}$ satisfies the desired universal property. First of all we have well defined functors

$$I_0 \colon \mathcal{C} \to \mathcal{A}$$
$$I_1 \colon \mathcal{D} \to \mathcal{A}$$

sending an arrow f to the one-element string (f) in $\mathcal{C} * \mathcal{D}$. If $\mathcal{A}$ is any other O-category and $F_0 \colon \mathcal{C} \to \mathcal{A}$, $F_1 \colon \mathcal{D} \to \mathcal{A}$ are functors in O-**Cat**, define

$$(F_0 * F_1) \colon \mathcal{C} * \mathcal{D} \to \mathcal{A}$$

as the functor sending a string $(f_n, \dots, f_0)$ where, for example, $f_0 \in \mathrm{Arr}(\mathcal{C})$ in

$$F_{\overline{n}}(f_n) \circ F_{\overline{n-1}}(f_{n-1}) \circ \dots \circ F_0(f_0)$$

where $\overline{k}$ denotes the residue of k in $\mathbb{Z}/2\mathbb{Z}$. It is easily checked that this gives a well-defined functor satisfying

$$F_0 * F_1 \circ I_k = F_k, \quad k = 0, 1$$

Moreover, the way we defined the arrows in $\mathcal{C} * \mathcal{D}$ implies that $F_0 * F_1$ is the only functor of O-categories satisfying the previous equations. It follows the thesis. □

Free O-categories

Denote by O-**Grph** the category of oriented graphs with set of vertexes equal to O and whose morphism are morphisms of graphs preserving the vertexes. We have a natural forgetful functor

$$\mathcal{U} \colon O\text{-}\mathbf{Cat} \to O\text{-}\mathbf{Grph}$$

We can construct a left adjoint

$$F \colon O\text{-}\mathbf{Grph} \to O\text{-}\mathbf{Cat}$$

sending a O-graph to the associated free category. The procedure is the same as for standard categories and we won't recall it here (see [Mac71, Ch. II.7]). We just remark that the same bound employed in the proof of previous lemma can be used to show that the free category over a O-graph yields a $\mathbb{U}$-small category, and hence it provides a well-defined functor. Checking the adjunction relation is then straightforward (we omit the details).

Definition 2.2.3. An O-category $\mathcal{C}$ is said to be a free category if it is in the essential image of the functor F.

Lemma 2.2.4. *The coproduct of free categories is again free.*

Proof. First of all observe that the category O-**Grph** admits coproducts: if $G, H \in \mathrm{Ob}(O\text{-}\mathbf{Grph})$ are O-graphs, define $G \sqcup H$ as the graph whose vertexes are the elements of O and whose arrows from $x \in O$ to $y \in O$ are

$$\mathrm{Arr}_G(x,y) \sqcup \mathrm{Arr}_H(x,y)$$

The verification is straightforward. At this point, the statement follows by nonsense: F commutes with colimits being a left adjoint, so that if $\mathcal{C} = F(G)$ and $\mathcal{D} = F(H)$, then

$$F(G \sqcup H) = F(G) * F(H) = \mathcal{C} * \mathcal{D}$$

proving that $\mathcal{C} * \mathcal{D}$ is a free category. □

Simplicial O-categories

Definition 2.2.5. A simplicial O-category is a simplicial object in the category O-**Cat**. The category of simplicial O-categories will be denoted by sO-**Cat**.

Remark 2.2.6. Observe that the "rigidity" of O-**Cat** implies that that a simplicial O-category is simply a category enriched over **sSet** with set of objects equal to O. Observe also that in general a simplicial object in **Cat** is a category enriched over **sSet** if and only if face and degeneracy functors are the identities over the objects.

Theorem B.2.9 shows that sO-**Cat** is a simplicial category. To give this category a model structure, we might hope to use Theorem B.2.15. In order to do so, we will need to analyze the effective epimorphisms in O-**Cat**.

Lemma 2.2.7. *The forgetful functor $\mathcal{U}\colon O$-**Cat** $\to O$-**Grph** creates reflexive coequalizers.*

Proof. Let

$$F, G\colon \mathcal{R} \to \mathcal{C}$$

be morphisms in O-**Cat** with a common (strict) section

$$S\colon \mathcal{C} \to \mathcal{R}$$

Introduce the following relation on $\mathrm{Arr}(\mathcal{C})$:

$$f \sim g \iff f = F(h),\ g = G(h) \text{ for some } h \in \mathrm{Arr}(\mathcal{R})$$

It's clear that this relation is symmetric and transitive. The existence of the section makes also clear that this relation is reflexive; finally, functoriality of F and G show immediately that this relation is compatible with compositions. As consequence, the quotient graph $\mathcal{D} := \mathcal{U}(\mathcal{C})/\sim$ has a natural structure of O-category. The universal property of the quotient is completely straightforward: if in the diagram

$$\begin{tikzcd} \mathcal{R} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \mathcal{C} \arrow[r, "P"] \arrow[dr, "Q"'] & \mathcal{D} \arrow[d, dotted, "\overline{P}"] \\ & & \mathcal{B} \end{tikzcd}$$

the functor Q coequalizes F and G, we immediately obtain a unique morphisms of O-graphs $\overline{P}\colon \mathcal{D} \to \mathcal{B}$. Since P is surjective on arrows, it is straightforward to check that $\overline{P}$ is a functor of O-categories. At this point, uniqueness is obvious. □

Corollary 2.2.8. *The category O-**Cat** is cocomplete.*

Proof. We already know that O-**Cat** has arbitrary coproducts; moreover, Lemma 2.2.7 implies that O-**Cat** has reflexive coequalizers (because O-**Grph** is obviously cocomplete). Abstract nonsense shows then that O-**Cat** has all the coequalizers. In fact, if we are given a pair of parallel arrows in a category $\mathcal{C}$ with coproducts and reflexive coequalizers

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

we obtain a reflexive diagram

$$\begin{tikzcd} X \sqcup Y \arrow[r, shift left, "f \sqcup 1"] \arrow[r, shift right, "g \sqcup 1"'] & Y \end{tikzcd}$$

It is straightforward to check that the coequalizer of this last diagram is also a coequalizer for the first diagram. □

Corollary 2.2.9. *The forgetful functor $\mathcal{U}\colon O$-**Cat** $\to O$-**Grph** preserves effective epimorphisms.*

Proof. Every effective epimorphism is in particular a reflexive coequalizer. □

Corollary 2.2.10. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an effective epimorphism in O-**Cat**. Then F is surjective on arrows.*

Proof. This follows from Corollary 2.2.9 and the fact (which we are going to prove) that every epimorphism in O-**Grph** is surjective on arrows. In fact, let $f\colon G_1 \to G_2$ be an epimorphism of O-graphs and let $\alpha\colon x \to y$ be an edge in G_2. Consider the O-graph G which has exactly one edge between a and b if $(a,b) \neq (x,y)$, and exactly two edges γ_0, γ_1 if $(a,b) = (x,y)$. Define $p\colon G_2 \to G$ as the only morphism sending every edge from x to y into γ_0 and $q\colon G_2 \to G$ as the only morphism sending α into γ_1 and every other edge from x to y into γ_0. Then $p \circ f = q \circ f$ if and only if α is not in the image of f. Since f is an epimorphism, this equality would imply $p = q$, which is impossible. It follows that f is surjective on arrows. □

Corollary 2.2.11. *Let $X, Y \in O$ be distinct objects and let $\mathcal{G}_{X,Y}$ be the O-graph characterized by the following condition:*

$$\mathrm{Edges}(A,B) = \begin{cases} \{*\} & \text{if } A = B \text{ or } (A,B) = (X,Y) \\ \emptyset & \text{otherwise.} \end{cases}$$

*Then $\mathcal{C}_{X,Y} := F(\mathcal{G}_{X,Y})$ is a projective object in O-**Cat**.*

Proof. Let $f\colon \mathcal{C} \to \mathcal{D}$ be an effective epimorphism. Corollary 2.2.10 implies that f is surjective on arrows. Since the following lifting problems

$$\begin{tikzcd} & \mathcal{C} \arrow[d, "f"] \\ \mathcal{C}_{X,Y} \arrow[r] \arrow[ur, dotted] & \mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{U}(\mathcal{C}) \arrow[d, "\mathcal{U}(f)"] \\ \mathcal{G}_{X,Y} \arrow[r] \arrow[ur, dotted] & \mathcal{U}(\mathcal{D}) \end{tikzcd}$$

are equivalent by adjoint nonsense, we conclude the proof. □

Corollary 2.2.12. *The set $S := \{\mathcal{C}_{X,Y}\}_{(X,Y)\in O^2}$ is a set of small projective generators for O-**Cat**. Moreover, for every other O-category $\mathcal{C}$ there is an effective epimorphism*

$$\coprod_{k \in I} \mathcal{C}_{X_k, Y_k} \to \mathcal{C}$$

Proof. Every object in S is a projective object thanks to Corollary 2.2.9; each $\mathcal{C}_{X,Y}$ is manifestly small. It is also clear that S is a set of generators. Finally, observe that every free category can be obtained as coproduct of the categories $\mathcal{C}_{X,Y}$. Write $\perp := F \circ \mathcal{U}$; the usual simplicial techniques show the existence of a reflexive diagram:

$$\perp^2\mathcal{C} \rightrightarrows \perp\mathcal{C} \longrightarrow \mathcal{C} \tag{2.8}$$

Applying the forgetful functor $\mathcal{U}$, we see that this diagram becomes a split coequalizer in O-**Grph**. Since $\mathcal{U}$ creates reflexive coequalizers, it also reflects them. It follows that the diagram (2.8) is a coequalizer in O-**Cat**. In particular, the map $\perp\mathcal{C} \to \mathcal{C}$ is a regular epimorphism, hence an effective epimorphism (here we are using the fact that O-**Cat** has binary products, as shown in Lemma 2.2.1). □

Theorem 2.2.13. *The category sO-**Cat** has a simplicial model category structure where*

1. *a functor $F\colon \mathcal{A} \to \mathcal{B}$ is a weak equivalence if and only if for each pair of objects $X, Y \in O$ the induced map*

$$\mathrm{Hom}_{\mathcal{A}}(X,Y;\mathbf{sSet}) \to \mathrm{Hom}_{\mathcal{B}}(X,Y;\mathbf{sSet})$$

is a weak equivalence of simplicial sets;

2. *a functor $F\colon \mathcal{A} \to \mathcal{B}$ is a fibration if for each pair of objects $X, Y \in O$ the induced map*

$$\mathrm{Hom}_{\mathcal{A}}(X,Y;\mathbf{sSet}) \to \mathrm{Hom}_{\mathcal{B}}(X,Y;\mathbf{sSet})$$

is a fibration;

3. *a map is a cofibration if and only if it has the LLP with respect to trivial fibrations.*

Proof. Theorem B.2.15 guarantees the existence of a model structure where a functor $F\colon \mathcal{A} \to \mathcal{B}$ is a weak equivalence or a fibration if and only if for each projective object $\mathcal{C}$, the induced map $\mathrm{Hom}_{\mathrm{s}O\text{-}\mathbf{Cat}}(\mathcal{C}, F; \mathbf{sSet})$ is a weak equivalence or a fibration. Proposition B.2.16 and Corollary 2.2.12 allow us to take $\mathcal{C} = \mathcal{C}_{X,Y}$. In this case, the map $\mathrm{Hom}_{\mathrm{s}O\text{-}\mathbf{Cat}}(\mathcal{C}, F; \mathbf{sSet})$ coincides exactly with the induced map

$$\mathrm{Hom}_{\mathcal{A}}(X, Y; \mathbf{sSet}) \to \mathrm{Hom}_{\mathcal{B}}(X, Y; \mathbf{sSet})$$

The theorem follows. □

Following Quillen and Dwyer-Kan we obtain the following characterization of cofibrations in sO-**Cat**. Recall first of all the following definition:

Definition 2.2.14. A map $f\colon A \to B$ in a category $\mathcal{C}$ is a strong retract of a map $g\colon A \to B'$ if there exists a commutative diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"] & A \arrow[r, equal] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r, "r"] & B' \arrow[r, "s"] & B \end{tikzcd}$$

where $sr = \mathrm{id}_B$.

Theorem 2.2.15. *A map in* sO-**Cat** *is a cofibration if and only if it is a strong retract of a free map. In particular, an object in* sO-**Cat** *is a cofibrant if and only if it is a retract of a free one.*

2.2.2 Dwyer-Kan localization

The general techniques exposed in [Wei94, Ch. 8.6] allows to use the adjunction $F \dashv U$ to construct an explicit cofibrant replacement functor. In fact, let $(\perp, \varepsilon, \delta)$ be the comonad associated to this adjunction. For an O-category $\mathcal{C}$ we define

$$\perp_k \mathcal{C} := \perp^{k+1} \mathcal{C}$$

and

$$d_i := \perp^i \varepsilon \perp^{k-i} \colon \perp_k \mathcal{C} \to \perp_{k-1} \mathcal{C}$$
$$s_i := \perp^i \delta \perp^{k-i} \colon \perp_k \mathcal{C} \to \perp_{k+1} \mathcal{C}$$

as face and degeneracy functors.

We get therefore a functor

$$\mathscr{F}\colon O\text{-}\mathbf{Cat} \to \mathrm{s}O\text{-}\mathbf{Cat}$$

defined by

$$\mathscr{F}(\mathcal{C}) := \{\perp_* \mathcal{C}\}$$

Reviewing $\mathcal{C}$ as a simplicial O-category concentrated in degree zero we obtain a natural functor

$$\mathscr{F}(\mathcal{C}) \to \mathcal{C}$$

which is a weak equivalence. At this point we can give the following

Definition 2.2.16. Let $\mathcal{C}$ be an O-category and let $W \subset \mathcal{C}$ be a subcategory. The standard simplicial localization (or Dwyer-Kan localization) of $\mathcal{C}$ is the category

$$L(\mathcal{C}, W) := \mathscr{F}(\mathcal{C})[\mathscr{F}(W)^{-1}]$$

We immediately obtain the following

Proposition 2.2.17. *Let $\mathcal{C}$ be an O-category and let $W \subset \mathcal{C}$ be a subcategory. Then we have a canonical isomorphism*

$$\pi_0 L(\mathcal{C}, W) = C[W^{-1}]$$

Remark 2.2.18. If in the definition of the Instead of taking a free resolution we could have taken any cofibrant resolution. It can be proven that the final result is invariant modulo homotopy in the category sO-**Cat**.

Remark 2.2.19. Those already knowing the construction of mapping spaces in a general model category will recognize a more global approach to the problem: instead of resolving an object at time, we are resolving the whole category. The advantage of this approach is that it produces mapping spaces which are composable. The lack of this property is, in fact, the main drawback of the other approach.

2.2.3 Homotopy (co)limits

There is an important relationship between the definition we gave of (co)limit (Definition 2.1.48) and homotopy (co)limits in model categories. The main result is the following:

Theorem 2.2.20. *Let $\mathcal{C}$ and $\mathcal{J}$ be fibrant simplicial categories and $F\colon \mathcal{J} \to \mathcal{C}$ a simplicial functor. Suppose we are given an object $C \in \mathrm{Ob}(\mathcal{C})$ and a compatible family of maps $\{\eta_I \colon F(I) \to C\}_{I \in \mathcal{J}}$. The following conditions are equivalent:*

1. *the maps η_I exhibit C as a homotopy colimit of the diagram F;*

2. *let $f\colon N(\mathcal{J}) \to N(\mathcal{C})$ be the simplicial nerve of F and $\overline{f}\colon N(\mathcal{J})^{\triangleright} \to N(\mathcal{C})$ the extension of f determined by the maps $\{\eta_I\}$. Then $\overline{f}$ is a colimit diagram in $N(\mathcal{C})$.*

Proof. See [HTT, Theorem 4.2.4.1]. □

In particular if $\mathcal{M}$ is a model category, its Dwyer-Kan localization is a fibrant simplicial category, and the localization preserves all the homotopical informations. It follows that if an ∞-category is presented by a model category, to compute colimits in the sense of Definition 2.1.48 we can compute them as homotopy colimits in $\mathcal{M}$. This will be useful in the next section.

2.3 Stable $(\infty, 1)$-categories

2.3.1 Definition

Definition 2.3.1. A pointed quasicategory S is a category endowed with an object 0 which is both initial and final.

Definition 2.3.2. Let S be a pointed quasicategory. A triangle in S is a morphism of simplicial sets from $\Delta^1 \times \Delta^1$ to S such that the image of $d^0 \times d^1 \colon \Delta^0 \to \Delta^1 \times \Delta^1$ is a zero object for S.

We will depict a triangle in a quasicategory as

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

Definition 2.3.3. Let S be a pointed quasicategory. A triangle in S is a fiber sequence if it is a pullback. Dually, a triangle is a cofiber sequence if it is a pushout.

Definition 2.3.4. A quasicategory S is said to be *stable* if the following conditions are satisfied:

1. S is pointed;

2. every morphism has a fiber and a cofiber;

3. a triangle is a fiber sequence if and only if it is a cofiber sequence;

2.3.2 Suspension and loop functors

A basic result concerning stable ∞-categories is that the their homotopy category is triangulated. This is the counterpart of a well known result of Quillen in the context of model categories (a nice exposition is given in [Hov99, Chapter 7]). We won't give the full proof of this theorem, even though it is not hard, because it has little to do with the main topic of this mémoire. However, we will show how to construct the suspension functor and its adjoint, the loop functor.

Let S be an ∞-category. We consider the full subcategory $\mathfrak{M}^\Sigma$ of $\mathcal{C} := \mathrm{Fun}(\Delta^1 \times \Delta^1, S)$ spanned by those pushout diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & Y \end{tikzcd} \tag{2.9}$$

where 0 and $0'$ are zero objects of S. Evaluation at the first vertex produces a natural functor

$$e_0 \colon \mathfrak{M}^\Sigma \to S$$

which is a Kan fibration. If moreover every morphism has a cofiber, then we see that the fibers of e_0 are nonempty and contractible. Lemma 2.1.44 implies that θ is a trivial Kan fibration; choose a section of θ

$$s \colon S \to \mathfrak{M}^\Sigma$$

and let $e_1 \colon \mathfrak{M}^\Sigma \to S$ be the evaluation at the last vertex. The composition $e_1 \circ s$ produces a functor

$$\Sigma \colon S \to S$$

which is called the suspension of S. In a dual way, we can define the loop functor $\Omega \colon S \to S$, using the category $\mathfrak{M}^\Omega$ whose objects are pullback diagrams of the same shape as above.

Remark 2.3.5. We would like to give an intuitive explanation for the unexperienced reader. Avoiding the technicalities of simplicial sets, we can imagine to work within a model category. Then, according to Theorem 2.2.20, we have to consider homotopy pushout diagrams; the construction given above defines the suspension of an object X in $\mathcal{C}$ to be the homotopy pushout of the diagram $* \leftarrow X \rightarrow *$, and this matches with the topological intuition of a homotopy pushout.

Proposition 2.3.6. *Let S be a pointed ∞-category and assume that every morphism in S has a fiber and a cofiber. Then Σ is left adjoint to Ω.*

Proof. Let $i \colon \mathfrak{M}^\Sigma \to \mathcal{C}$ and $j \colon \mathfrak{M}^\Omega \to \mathcal{C}$ be the natural inclusions. Since $\mathfrak{M}^\Sigma$ and $\mathfrak{M}^\Omega$ are ∞-categories, it follows that i and j are inner fibrations. Set

$$\mathfrak{M} := \mathfrak{M}^\Sigma{}_i\star_j \mathfrak{M}^\Omega$$

Since $\mathcal{C}$ is a quasicategory, Proposition A.3.9 implies that $\mathfrak{M}$ is an ∞-category. Moreover Proposition A.3.10 produces a morphism of simplicial sets $p \colon \mathfrak{M} \to \Delta^1$ such that

$$\mathfrak{M}_{\{0\}} \simeq \mathfrak{M}^\Sigma, \qquad \mathfrak{M}_{\{1\}} \simeq \mathfrak{M}^\Omega$$

We claim that p is both cartesian and cocartesian. Let us show the cartesian statement; the other will be similar. If an object $\xi = \xi_n$ of $\mathfrak{M}$ lies over 1, then it represents a pullback square of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & Y \end{tikzcd}$$

Let ξ_{n-1}

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & Y \end{tikzcd}$$

be the cofiber of the morphism $0 \to Y$. We obtain a morphism (unique up to a contractible space of choices) $\xi_{n-1} \to \xi_n$ in $\mathcal{C}$,[2] which corresponds to an edge $f\colon \Delta^1 \to \mathfrak{M}$ from ξ_{n-1} to ξ_n. We claim that f is a cartesian morphism. Accordingly to Lemma 2.1.58 we have to show that every lifting problem of the following form

$$\begin{tikzcd}
\Delta^1 \arrow[d, "{[n-1,n]}"'] \arrow[dr, "f"] & \\
\Lambda^n_n \arrow[r, "\alpha"] \arrow[d, tail] & \mathfrak{M} \arrow[d, "p"] \\
\Delta^n \arrow[r] \arrow[ur, dotted] & \Delta^1
\end{tikzcd}$$

has solution for every $n \geq 2$. However, the commutativity of the diagram forces the map $\Delta^n \to \Delta^1$ to be the one induced by the map $\varphi\colon \mathbf{n} \to \mathbf{1}$ characterized by $\varphi^{-1}(1) = \{n\}$.

Consider the inclusion

$$\iota\colon \partial\Delta^{n-1} \longrightarrow \Delta^{n-1} \xrightarrow{d^n} \Delta^n$$

and let $\overline{\alpha} := \alpha \circ \iota$. The definition of n-simplexes in the relative join $\mathfrak{M} = \mathfrak{M}^{\Sigma}{}_{i}{\star}_{j}\, \mathfrak{M}^{\Omega}$ shows that our problem is equivalent to the induced lifting problem:

$$\begin{tikzcd}
\partial\Delta^{n-1} \arrow[r, "\overline{\alpha}"] \arrow[d, tail] & \mathfrak{M} \\
\Delta^{n-1} \arrow[ur, dotted] &
\end{tikzcd}$$

Let $e_0\colon \mathfrak{M} \to S$ be the evaluation at the initial vertex. If we denote by $\beta\colon \Delta^{n-1} \to \mathfrak{M}$ the $(n-1)$-simplex corresponding via α to the inclusion of the $(n-1)$-th face of Λ^n_n we obtain an $(n-1)$-simplex ω in S. Since

$$e_0(\xi_n) = e_0(\xi_{n-1})$$

by construction, we obtain a commutative diagram

$$\begin{tikzcd}
\partial\Delta^{n-1} \arrow[r, "\overline{\alpha}"] \arrow[d, tail] & \mathfrak{M} \arrow[d, "e_0"] \\
\Delta^{n-1} \arrow[r, "\omega"] \arrow[ur, dotted] & S
\end{tikzcd}$$

Since e_0 is a trivial Kan fibration, the lifting exists, and the proof is complete. □

If S is stable, then $\mathcal{M}^{\Sigma} = \mathcal{M}^{\Omega}$, and Σ and Ω define an adjoint equivalence of ∞-categories. We conclude citing the following result:

Theorem 2.3.7. *Let S be stable ∞-category. Then $\mathrm{h}S$ is triangulated and the translation functor is exactly the suspension Σ.*

Proof. See [HA, Theorem 1.1.2.14]. □

2.4 Spectra

In this last section we introduce the idea of spectrum, which is in a certain sense the higher algebraic analogue of an abelian group.

[2] This is essentially a consequence of Corollary 2.1.47, jointly with our definition of limit.

Notation 2.4.1. We let $\mathcal{S}$ denote the ∞-category of (small) spaces. This can be obtained as follows: one start with a Grothendieck universe $\mathbb{U}$ and consider the model category $\mathbb{U}$-**sSet** of $\mathbb{U}$-simplicial sets, with the standard model structure. Taking the Dwyer-Kan localization and successively the simplicial nerve, we obtain an ∞-category which is $\mathbb{V}$-small, where $\mathbb{V}$ is a Grothendieck universe extending $\mathbb{U}$.

We will denote by $\mathcal{S}_*$ the category of pointed objects of $\mathcal{S}$. Finally, we will denote by $\mathcal{S}^{\text{fin}}$ the smallest full subcategory of $\mathcal{S}$ which contains the final object $*$ and is closed under finite colimits.

Definition 2.4.2. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories.

1. if $\mathcal{C}$ admits pushouts, F is said to be *excisive* if it carries pushout squares in $\mathcal{C}$ to pullback squares in $\mathcal{D}$;

2. if $\mathcal{C}$ admits a final object $*$, F is said to be *reduced* if $F(*)$ is a final object of $\mathcal{D}$

In the particular case where $\mathcal{D}$ has a zero object, we give a special name to those functors which are both excisive and reduced:

Definition 2.4.3. Let $\mathcal{D}$ be a pointed ∞-category. A functor $F \colon \mathcal{C} \to \mathcal{D}$ is said to be *strongly excisive* if it is reduced and excisive.

We are finally ready to give the definition of spectrum object:

Definition 2.4.4. Let $\mathcal{C}$ be a pointed ∞-category. A *spectrum* in $\mathcal{C}$ is a strongly excisive functor $\mathcal{S}^{\text{fin}}_* \to \mathcal{C}$.

This definition is particularly elegant, and it allows to deduce immediately that given a pointed ∞-category $\mathcal{C}$, there exists an ∞-category whose objects are the spectra of $\mathcal{C}$ (namely, the full subcategory of $\mathrm{Fun}(\mathcal{S}^{\text{fin}}_*, \mathcal{C})$ spanned by the strongly excisive functors). However, to produce in practice spectra, it is useful to have another characterization of them.

Proposition 2.4.5. *Let $\mathcal{C}$ be a pointed ∞-category with finite colimits. A spectrum $F \colon \mathcal{S}^{\text{fin}}_* \to \mathcal{C}$ is equivalently determined by a sequence $\{E_n\}_{n\in\mathbb{N}}$ of objects in $\mathcal{C}$ equipped with equivalences*

$$E_n \simeq \Omega E_{n+1}$$

Proof. Given a sequence $E := \{E_n\}_{n\in\mathbb{N}}$ with equivalences $E_n \simeq \Omega E_{n+1}$, define

$$F_E(S^n) := E_n$$

Since the spheres generates $\mathcal{S}^{\text{fin}}_*$ under finite colimits, we obtain that there is a Kan extension of F_E to the whole $\mathcal{S}^{\text{fin}}_*$; moreover, this extension is unique up to a contractible space of choices.

Conversely, given a spectrum $F \colon \mathcal{S}^{\text{fin}}_* \to \mathcal{C}$, define

$$E_n := F(S^n)$$

Since $S^{n+1} \simeq \Sigma S^n$, using the fact that F is strongly excisive, we obtain

$$E_n = F(S^n) \simeq \Omega F(S^{n+1}) = \Omega E_{n+1}$$

This is enough to conclude. $\square$

Chapter 3

Homotopy theory of algebras

3.1 Differential graded modules

3.1.1 Graded modules

We will consider $\mathbb{Z}$ as a discrete category. We moreover fix a commutative ring (with unit) A.

Definition 3.1.1. The category of graded A-modules is an element of the functor category $A\text{-}\mathbf{GMod} := (A\text{-}\mathbf{Mod})^{\mathbb{Z}}$. We will denote by $M_* = \{M_n\}_{n\in\mathbb{Z}}$ an object in this category.

Remark 3.1.2. More classically a differential graded A-module is an A-module M with a direct sum decomposition

$$M = \bigoplus_{n\in\mathbb{Z}} M_n \tag{3.1}$$

Under this identification, the morphisms of the category $A\text{-}\mathbf{GMod}$ becomes simply morphisms of A-modules $f\colon M \to N$ such that

$$f(M_i) \subset f(N_i)$$

Notation 3.1.3. If M_* is a differential graded A-module and we are given a direct sum decomposition as in (3.1), an element of M lying in M_n for some $n \in \mathbb{Z}$ is said to be homogeneous of degree n; if x is such an element, we will express this writing $|x| = n$.

Remark 3.1.4. Recall that if $\mathcal{A}$ is an abelian category and $\mathbf{I}$ is a small category, the functor category $\mathcal{A}^{\mathbf{I}}$ is again an abelian category. In particular, we see that $A\text{-}\mathbf{GMod}$ is an abelian category. In particular $\mathbf{G} := \mathbf{Funct}(\mathbb{Z}, -)$ defines a functor which goes from the category of (small) abelian categories to (small) abelian categories.

For every $n \in \mathbb{Z}$ there exists a forgetful (additive) functor

$$p_n\colon A\text{-}\mathbf{GMod} \to A\text{-}\mathbf{Mod}$$

sending $M = \{M_n\}_{n\in\mathbb{Z}}$ to M_n, which is trivially exact. This functor has a left adjoint. In fact, for every $m \in \mathbb{Z}$ we can consider

$$S^m\colon A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{GMod}$$

defined on objects by $M \mapsto S^m(M)$, where

$$(S^m(M))_n := \begin{cases} M & \text{if } n = -m \\ 0 & \text{otherwise} \end{cases}$$

Lemma 3.1.5. *We have the adjunction $S^{-n} \dashv p_n$.*

Proof. We have an obvious natural map

$$\eta_M\colon M \to S^{-n}(M_n)$$

and it is straightforward to check that this map is a universal arrow from M to the functor S^{-n}. $\square$

Notation 3.1.6. We will denote by $A[n]$ the graded module $S^n(A)$. If $n = 0$ we will write A instead of $A[0]$ whenever it is clear that we are working inside A-**GMod**.

Symmetric monoidal closed structure

It is possible to endow A-**GMod** with a symmetric monoidal closed structure. We begin with a definition of the tensor product of graded A-modules. Using the functor

$$\mathbf{G} := \mathbf{Funct}(\mathbb{Z}, -)$$

we observe that we can use the ordinary product of A-modules to obtain an additive functor

$$- \boxtimes_A - \colon \mathbf{G}(A\text{-}\mathbf{Mod}) \times G(A\text{-}\mathbf{Mod}) \to \mathbf{G}^2(A\text{-}\mathbf{Mod}) \simeq \mathbf{Funct}(\mathbb{Z} \times \mathbb{Z}, A\text{-}\mathbf{Mod})$$

defined on objects as

$$(\{M_p\}_{p \in Z}, \{N_q\}_{q \in Z}) \mapsto \{M_p \otimes N_q\}_{(p,q) \in \mathbb{Z} \times \mathbb{Z}}$$

Moreover, we have a functor

$$\mathrm{Tot}^{\oplus} \colon \mathbf{G}^2(A\text{-}\mathbf{Mod}) \to \mathbf{G}(A\text{-}\mathbf{Mod})$$

defined by

$$\mathrm{Tot}^{\oplus}(M_{*,*}) := \left\{ \bigoplus_{p+q=n} M_{p,q} \right\}_{n \in \mathbb{Z}}$$

It is easily checked that this functor is additive.

Definition 3.1.7. The tensor product of graded A-modules is by definition the additive functor

$$- \otimes_A - := \mathrm{Tot}^{\oplus}(- \boxtimes_A -) \colon A\text{-}\mathbf{GMod} \times A\text{-}\mathbf{GMod} \to A\text{-}\mathbf{GMod}$$

Proposition 3.1.8. $(A\text{-}\mathbf{GMod}, \otimes_A, A)$ *is a symmetric monoidal category, and the symmetry isomorphism can be chosen as the natural transformation*

$$\sigma_{M_*, N_*} \colon M_* \otimes_A N_* \to N_* \otimes_A M_*$$

induced by

$$M_p \otimes_A N_q \to N_q \otimes M_p \colon m \otimes_A n \mapsto (-1)^{pq} n \otimes_A m$$

Proof. It is easily checked the existence of natural isomorphisms

$$- \otimes_A A \simeq \mathrm{Id}_{A\text{-}\mathbf{GMod}} \simeq A \otimes_A -$$

To prove the existence of the associativity isomorphism let us observe that the universal property of direct sums implies the existence of natural isomorphisms

$$\bigoplus_{r \in \mathbb{Z}} \bigoplus_{p+q=n-r} M_{p,q,r} \simeq \bigoplus_{p+q+r=n} M_{p,q,r} \simeq \bigoplus_{p \in \mathbb{Z}} \bigoplus_{q+r=n-p} M_{p,q,r}$$

This implies immediately that

$$\mathrm{Tot}^{\oplus} \circ \mathrm{Tot}^{\oplus}_{2,3} \simeq \mathrm{Tot}^{\oplus} \circ \mathrm{Tot}^{\oplus}_{1,2}$$

Moreover, associativity of tensor product shows as well that

$$(M_* \boxtimes_A N_*) \boxtimes_A P_* \simeq M_* \boxtimes_A (N_* \boxtimes_A P_*)$$

It follows the existence of a natural isomorphism

$$(M_* \otimes_A N_*) \otimes_A P_* \simeq M_* \otimes_A (N_* \otimes_A P_*)$$

It is a straightforward exercise to check that the pentagonal axiom holds.

Finally, the map σ defines indeed a natural isomorphism of $- \otimes_A -$, and it is easy to verify that the additional diagrams relating associativity and symmetry are commutative. $\square$

Notation 3.1.9. Let M_* be a graded A-module. We will denote by $M_*[n]$ the tensor product

$$M_*[n] := M_* \otimes_A A[n]$$

We will say that $M_*[n]$ is obtained shifting M_* by n.

Remark 3.1.10. Inspection reveals immediately that

$$(M_*[n])_m = M_{m-n}$$

Lemma 3.1.11. *The shifting functor* $- \otimes_A A[n]$ *is left adjoint to* $- \otimes_A A[-n]$.

Proof. In fact the elements of $\mathrm{Hom}_{A\text{-}\mathbf{GMod}}(M_*[-n], N_*)$ are canonically identified with A-linear maps

$$f\colon \bigoplus_{p\in\mathbb{Z}} M_p \to \bigoplus_{q\in\mathbb{Z}} N_q$$

such that

$$f(M_p) \subset N_{q+n}$$

On the other side, the elements of $\mathrm{Hom}_{A\text{-}\mathbf{GMod}}(M_*, N_*[n])$ have the same canonical description, so that the result follows. □

Definition 3.1.12. Let M_*, N_* be a graded A-modules. A map from M_* to N_* *of degree* n is an element of $\mathrm{Hom}_{A\text{-}\mathbf{GMod}}(M_*[-n], N_*)$.

We can now define the internal hom for the monoidal structure of A-**GMod**. Given graded A-modules M_* and N_* define

$$(\mathbf{Hom}_A(M_*, N_*))_n := \mathrm{Hom}_{A\text{-}\mathbf{GMod}}(M_*[-n], N_*)$$

Proposition 3.1.13. *For every graded* A*-module* N_* *it holds the adjunction relation*

$$- \otimes_A N_* \dashv \mathbf{Hom}_A(N_*, -)$$

Proof. Fix a graded A-module N_*. For every pair of integers $(p, q) \in \mathbb{Z}^2$ consider

$$\mathrm{ev}_{p,q}\colon \mathbf{Hom}_A(N_*, P_*)_p \otimes_A N_q \to P_{p+q}$$

defined by

$$\mathrm{ev}_{p,q}(f \otimes n) := f(n)$$

This is well defined because f is a map of degree p, i.e. $f(N_q) \subset P_{p+q}$. Now define

$$\mathrm{ev}\colon \mathbf{Hom}_A(N_*, P_*) \otimes_A N_* \to P_*$$

to be the sum of the morphisms $\mathrm{ev}_{p,q}$. Naturality in P_* is clear, so that we obtain a natural transformation

$$\mathrm{ev}\colon \mathbf{Hom}_A(N_*, -) \otimes_A N_* \to \mathrm{Id}_{A\text{-}\mathbf{GMod}}$$

We claim moreover that ev is a universal arrow from $- \otimes_A N_*$ to P_*. In fact, let M_* be any other graded A-module and let

$$\alpha\colon M_* \otimes_A N_* \to P_*$$

be any map. Define

$$\beta\colon M_* \to \mathbf{Hom}_A(N_*, P_*)$$

to be the sum of $\beta_p\colon M_p \to \mathbf{Hom}_A(N_*, P_*)_p$, where β_p is the map

$$\beta_p(m) := \alpha(m \otimes_A -)$$

Inspection reveals immediately that

$$\mathrm{ev} \circ (\beta \otimes \mathrm{id}_{N_*}) = \alpha$$

Finally, if $\gamma\colon M_* \to \mathbf{Hom}(N_*, P_*)$ is another map satisfying previous equation, we see that

$$\gamma(m)(n) = \alpha(z \otimes n)$$

for every $m \in M_p$ and $n \in N_q$. It follows that $\gamma = \beta$, so that the adjunction is completely proved. □

3.1.2 Differential graded modules

We now introduce a variation on the constructions done up to this moment. The proofs do not change very much, hence we will not give the details of them. Also in this section A denotes a fixed commutative ring (with unit).

Definition 3.1.14. A *differential graded A-module* (in short dg A-module) is a graded A-module M_* together with a map of degree -1

$$d\colon M_* \to M_*$$

such that $d^2 = 0$, called the *differential* of M_*. A morphism of dg A-modules is a morphism of graded A-modules commuting with the differentials.

Notation 3.1.15. Let (M_*, d) be a dg A-module. We will write d_n to denote the map $d_n\colon M_n \to M_{n-1}$ induced by d.

Differential graded A-modules and maps between them can be obviously organized into a category, which we will denote by $A\text{-}\mathbf{Mod}^{\mathrm{dg}}$ or with $\mathbf{Ch}(A\text{-}\mathbf{Mod})$.

Lemma 3.1.16. *$A\text{-}\mathbf{Mod}^{\mathrm{dg}}$ is an abelian category.*

Proof. It is a straightforward check. □

We have an obvious forgetful functor

$$U\colon A\text{-}\mathbf{Mod}^{\mathrm{dg}} \to A\text{-}\mathbf{GMod}$$

which forgets the differential. It is also possible to define a less trivial functor, given by the cohomologies of a differential graded module:

Definition 3.1.17. Let (M_*, d) be a dg A-module. Define

$$H^n(M_*, d) := \ker d_n / \operatorname{Im} d_{n+1}$$

Define moreover

$$H^*(M_*, d) := \{H^n(M_*, d)\}$$

It is straightforward to see that H^* defines a true functor

$$H^*\colon A\text{-}\mathbf{Mod}^{\mathrm{dg}} \to A\text{-}\mathbf{GMod}$$

Finally, let us observe that each graded A-module can be reviewed as a dg A-module with zero differential. In particular, the functor $S^n\colon A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{GMod}$ can be thought as a functor

$$S^n\colon A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod}^{\mathrm{dg}}$$

Symmetric monoidal structure

The functor $\mathrm{Tot}^{\oplus}\colon \mathbf{G}^2(A\text{-}\mathbf{Mod}) \to A\text{-}\mathbf{GMod}$ can be extended to a functor

$$\mathrm{Tot}^{\oplus}\colon \mathbf{Ch}^2(A\text{-}\mathbf{Mod}) \to \mathbf{Ch}(A\text{-}\mathbf{Mod})$$

Remark 3.1.18. An object in $\mathbf{Ch}^2(A\text{-}\mathbf{Mod})$ is a double complex $\{M_{p,q}, d^h_{p,q}, d^v_{p,q}\}$ where the maps

$$d^h_{p,q}\colon M_{p,q} \to M_{p-1,q}, \qquad d^v_{p,q}\colon M_{p,q} \to M_{p,q-1}$$

satisfy the relations

$$d^h_{p-1,q} \circ d^h_{p,q} = 0, \quad d^v_{p,q-1} \circ d^v_{p,q} = 0, \quad d^v_{p-1,q} \circ d^h_{p,q} = d^h_{p,q-1} \circ d^v_{p,q}$$

In order to define $\mathrm{Tot}^{\oplus}$ we simply have to take care of the differentials. We will define the differential of $\mathrm{Tot}^{\oplus}(M_{p,q}, d^h_{p,q}, d^v_{p,q})$ to be the sum of

$$d^h_{p,q} + (-1)^p d^v_{p,q} \colon M_{p,q} \to M_{p-1,q} \oplus M_{p,q-1}$$

It is a straightforward exercise to check that we obtain in this way a chain complex.

On the other side, it is straightforward to check that if (M_*, d) and (N_*, δ) are differential graded A-modules, then

$$(M_p \otimes_A N_q, d_p \otimes \mathrm{id}_{N_q}, \mathrm{id}_{M_p} \otimes \delta_q)$$

defines an element $M_* \boxtimes N_*$ in $\mathbf{Ch}^2(A\text{-}\mathbf{Mod})$. As we previously did for graded modules we set

$$M_* \otimes_A N_* := \mathrm{Tot}^{\oplus}(M_* \boxtimes N_*)$$

The following lemma holds as well:

Lemma 3.1.19. $(A\text{-}\mathbf{Mod}^{\mathrm{dg}}, \otimes_A, A)$ *is a symmetric monoidal category.*

Proof. Straightforward. □

As before, we can introduce an internal hom for this monoidal structure. Given cochain complexes (M_*, d) and (N_*, δ) we endow $\mathbf{Hom}_A(U(M_*), U(N_*))$ with the following differential ρ: if $f \colon U(M_*) \to U(N_*)$ is a map of degree n, then

$$\rho(f) := \delta \circ f - (-1)^n f \circ d$$

It is an easy exercise to check that this defines a cochain complex structure on

$$\mathbf{Hom}_A(U(M_*), U(N_*))$$

We will denote this cochain complex as $\mathbf{Hom}_A^{\mathrm{dg}}(M_*, N_*)$.

Proposition 3.1.20. *For every cochain complex (N_*, d_N) the adjunction relation*

$$- \otimes_A N_* \dashv \mathbf{Hom}_A^{\mathrm{dg}}(N_*, -)$$

holds.

Proof. One simply has to check that under the adjunction of Proposition 3.1.13 maps of differential graded A-modules correspond to maps of differential graded A-modules, and this is straightforward. □

Example 3.1.21. Assume that N_* is concentrated in degree 0. Then for every other chain complex M_* we have directly from the definition:

$$(\mathbf{Hom}_A^{\mathrm{dg}}(M_*, N_*))_n = \mathrm{Hom}_A(M_{-n}, N)$$

Moreover, if $f \in \mathrm{Hom}_A(M_n, N)$ then

$$\rho(f) = (-1)^{n+1} f \circ d$$

Therefore, $\mathbf{Hom}_A^{\mathrm{dg}}(M_*, N_*)$ is isomorphic to the chain complex obtained by applying the contravariant functor $\mathrm{Hom}_A(-, N)$ to M_*. In fact, this follows from the following easy observation: if (M_*, d) is any chain complex and $n \in \mathbb{Z}$ is an integer, introduce a new differential $d^{(n)}$ on M_* defined by

$$d^{(n)}_m = \begin{cases} d_m & \text{if } m \neq n \\ -d_n & \text{if } m = n \end{cases}$$

Then we can define $f \colon (M_*, d) \to (M_*, d^{(n)})$ by setting

$$f_m := \begin{cases} \mathrm{id} & \text{if } m > n \\ -\mathrm{id} & \text{if } m \leq n \end{cases}$$

f is a morphism of chain complexes which is an isomorphism.

3.1.3 Model structure

In this last part concerning differential graded A-modules we discuss a model structure on $A\text{-}\mathbf{Mod}^{\mathrm{dg}}$. The result is very well known and it is proved in full detail in [Hov99, Chapter 2.3]. For this reason, we will omit the proof of the main theorem.

Definition 3.1.22. A morphism $f\colon M_* \to N_*$ of cochain complexes is said to be a *quasi–isomorphism* if the induced map $H^*(f)\colon H^*(M_*) \to H^*(N_*)$ is an isomorphism.

Definition 3.1.23. For every A-module M and every $n \in \mathbb{Z}$ define a cochain complex $(D^n(M), d)$ by setting:

$$(D^n(M))_p := \begin{cases} M & \text{if } p = n \text{ or } p = n+1 \\ 0 & \text{otherwise} \end{cases} \qquad d_p := \begin{cases} \mathrm{id}_M & \text{if } p = n \\ 0 & \text{otherwise} \end{cases}$$

Define I to be the set of all inclusions $S^n(A) \to D^n(A)$ and let J be the set of all the maps $0 \to D^n(A)$. The main result is the following:

Theorem 3.1.24. *There exists a cofibrantly generated model structure on $A\text{-}\mathbf{Mod}^{\mathrm{dg}}$ with I as set of generating cofibrations and J as set of generating trivial cofibrations, and where weak equivalences are exactly quasi–isomorphisms.*

Proof. See [Hov99, Theorem 2.3.11]. □

It is useful to characterize fibrations and cofibrations for this model structure:

Proposition 3.1.25. *In the model structure on $A\text{-}\mathbf{Mod}^{\mathrm{dg}}$ introduced in Theorem 3.1.24 the following characterization hold:*

1. *a map $f\colon X \to Y$ is a fibration if and only if it is degreewise surjective;*
2. *a cofibrant chain complex is degreewise projective;*
3. *a map $f\colon X \to Y$ is a cofibration if and only if it is a degreewise split inclusion with cofibrant cokernel.*

Proof. See [Hov99, Proposition 2.3.5, Lemma 2.3.6, Proposition 2.3.9]. □

Using the machinery of Dwyer-Kan localization we can pass from $A\text{-}\mathbf{Mod}^{\mathrm{dg}}$ to an ∞-category, which we will denote by

$$A\text{-}\mathrm{Mod}^{\mathrm{dg}}$$

If $A = k$ is a field we will also adopt the notations $\mathbf{Vect}_k^{\mathrm{dg}}$ and $\mathrm{Vect}_k^{\mathrm{dg}}$ to denote the categories $k\text{-}\mathbf{Mod}^{\mathrm{dg}}$ and $k\text{-}\mathrm{Mod}^{\mathrm{dg}}$.

3.2 (Commutative) differential graded algebras

The ultimate goal of this section is to endow the category of commutative differential graded algebras with a model structure. In fact, the general result [SS00, Theorem 4.1] does not apply because of the commutativity requirement; to explain the difficulty, let us say that the natural classes of fibrations and weak equivalences fail in certain situations (we will give an explicit counterexample later on). Therefore, we will need some additional hypothesis in order to apply Theorem B.2.1.

3.2.1 Definition

We begin by recalling the classical theory of (commutative) differential graded algebras. Let k be a fixed commutative ring.

Definition 3.2.1. A *differential graded k-algebra* is a monoid in the symmetric monoidal category $(k\text{-}\mathbf{Mod}^{\mathrm{dg}}, \otimes_k, k)$. A *commutative differential graded k-algebra* is a differential graded k-algebra equipped with a structure of commutative monoid.

Notation 3.2.2. We will denote by $\mathbf{Alg}_k^{\mathrm{dg}}$ and $\mathbf{CAlg}_k^{\mathrm{dg}}$ the categories of differential graded algebras and commutative differential graded algebras over k, respectively.

The categories $\mathbf{Alg}_k^{\mathrm{dg}}$ and $\mathbf{CAlg}_k^{\mathrm{dg}}$ comes equipped with forgetful functors

$$\mathcal{V}\colon \mathbf{Alg}_k^{\mathrm{dg}} \to \mathbf{Mod}_k^{\mathrm{dg}}, \qquad \mathcal{U}\colon \mathbf{CAlg}_k^{\mathrm{dg}} \to \mathbf{Mod}_k^{\mathrm{dg}}$$

Both these functors have a left adjoint, given by the tensor algebra and the symmetric (tensor) algebra:

$$\mathcal{T}\colon \mathbf{Mod}_k^{\mathrm{dg}} \to \mathbf{Alg}_k^{\mathrm{dg}}, \qquad \mathrm{Sym}\colon \mathbf{Mod}_k^{\mathrm{dg}} \to \mathbf{CAlg}_k^{\mathrm{dg}}$$

If M_* is a differential graded k-module, we define $\mathcal{T}(M_*)$ to be the differential graded algebra which in degree n is:

$$\mathcal{T}(M_*)_n := \bigoplus_{s\in\mathbb{Z}} \bigoplus_{i_1+\ldots+i_s=n} M_{i_1} \otimes_k \ldots \otimes_k M_{i_s}$$

The differential of $\mathcal{T}(M_*)$ is given by extending the differential of M_* using the (graded) Leibniz rule. The symmetric algebra $\mathrm{Sym}(M_*)$ is defined to be the quotient of $\mathcal{T}(M_*)$ by the two-sided ideal generated by those elements of the form

$$x \otimes y - (-1)^{|x||y|} y \otimes x$$

where x and y range over the homogeneous elements of M_*. It is a straightforward exercise to verify the following proposition:

Proposition 3.2.3. *There are adjoint relations $\mathcal{T} \dashv \mathcal{V}$ and $\mathrm{Sym} \dashv \mathcal{U}$.*

Proof. This is a straightforward verification. □

3.2.2 Model structure

In order to endow $\mathbf{CAlg}_k^{\mathrm{dg}}$ we wish to apply Theorem B.2.1 to the adjunction we previously introduced:

$$\mathrm{Sym}\colon \mathbf{Mod}_k^{\mathrm{dg}} \rightleftarrows \mathbf{CAlg}_k^{\mathrm{dg}} : \mathcal{U}$$

The category $\mathbf{Mod}_k^{\mathrm{dg}}$ is always cofibrantly generated (see Theorem 3.1.24), so that we have only to verify the following facts:

1. the functor $\mathcal{U}$ commutes with sequential colimits;
2. defining a map f to be a fibration or a weak equivalence if $\mathcal{U}(f)$ is so, then every map with the LLP with respect to every fibration is a weak equivalence.

The first statement has a completely formal proof and doesn't need any assumption on the base ring k. In fact, one has even a stronger result:

Lemma 3.2.4. *The forgetful functor $\mathcal{U}\colon \mathbf{CAlg}_k^{\mathrm{dg}} \to \mathbf{Mod}_k^{\mathrm{dg}}$ creates sifted colimits.*

Proof. Let $\mathbf{I}$ be a sifted category and let $F\colon \mathbf{I} \to \mathbf{CAlg}_k^{\mathrm{dg}}$ be an $\mathbf{I}$-indexed diagram. Set $G := \mathcal{U} \circ F$ and let

$$A := \mathrm{colim}_{i\in\mathbf{I}}\, G(i)$$

be the colimit computed in $\mathbf{Mod}_k^{\mathrm{dg}}$. Since the diagonal map $\mathbf{I} \to \mathbf{I}\times\mathbf{I}$ is final, it follows that

$$\mathrm{colim}_{i\in\mathbf{I}}\, G(i) \otimes_k \mathrm{colim}_{j\in\mathbf{I}}\, G(j) \simeq \lim_{(i,j)\in\mathbf{I}^2} G(i) \otimes_k G(j) \simeq \lim_{i\in\mathbf{I}} G(i) \otimes_k G(i)$$

The multiplication maps $\mu_i\colon G(i) \otimes_k G(i) \to G(i)$ then produce a map

$$\mu\colon A \otimes_k A \to A$$

and the universal property of the colimits allows to check that μ is a graded commutative multiplication, and that the natural derivation on A satisfies the graded Leibniz rule. The natural maps $F(i) \to A$ are then morphisms of graded commutative algebras by construction; it is straightforward to check that A is the colimit of F in $\mathbf{CAlg}_k^{\mathrm{dg}}$. □

It is much harder to verify condition 2.; in fact, it is impossible without some additional assumption on the base ring k as the following example shows.

Example 3.2.5. Let k be a field of characteristic 2 and let $D(n)$ be the chain complex with one copy of k in degrees n and $n-1$ (with identity boundary). Then $0 \to D(n)$ is a weak equivalence, but

$$k = \mathrm{Sym}(0) \to \mathrm{Sym}(D(n))$$

cannot be a weak equivalence. In fact, if y is a nonzero element of $D(n)$ in degree n, then $y^2 = y \otimes y \in \mathrm{Sym}(D(n))$ satisfies

$$d(y^2) = 2y \otimes d(y) = 0$$

so that it is a cycle which cannot be a boundary. If Theorem B.2.1 could be applied, then adjoint nonsense would show that Sym takes acyclic cofibrations to acyclic cofibrations, which is not the case (this is [GS06, Example 3.7]).

In order to rule out pathologies like in previous example, we will assume from this moment on that k is a *field of characteristic* 0. Under this assumption, we can prove the following result, which is really strong:

Lemma 3.2.6. *Let k be a field of characteristic 0. If a morphism $f\colon V_* \to W_*$ in $\mathbf{Mod}_k^{\mathrm{dg}}$ is a weak equivalence then $\mathcal{U}(\mathrm{Sym}(f))$ is a weak equivalence as well.*

Proof. Assume that $f\colon V_* \to W_*$ is a weak equivalence. Since in $\mathbf{Vect}_k^{\mathrm{dg}}$ every object is both fibrant and cofibrant (cfr. Proposition 3.1.25), we see that f is a chain equivalence. We are therefore reduced to show that if $f\colon V_* \to W_*$ is nullhomotopic, then $\mathrm{Sym}(f)$ is nullhomotopic as well.

Observe that if $f\colon V_* \to W_*$ is a chain equivalence, then the induced map

$$f^{\otimes n}\colon V_*^{\otimes n} \to W_*^{\otimes n}$$

is a chain equivalence for every $n \geq 0$. Since k is of characteristic 0, we have a retraction

$$\psi_n\colon \mathrm{Sym}^n(V_*) \to V_*^{\otimes n}$$

defined by

$$\psi_n(v_1 \cdots v_n) := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}$$

This implies the thesis, since we have a retraction diagram

$$\begin{tikzcd}
\mathrm{Sym}^n(V_*) \arrow[r] \arrow[d, "\mathrm{Sym}^n(f)"] & V_*^{\otimes n} \arrow[r] \arrow[d, "f^{\otimes n}"] & \mathrm{Sym}^n(V_*) \arrow[d, "\mathrm{Sym}^n(f)"] \\
\mathrm{Sym}^n(W_*) \arrow[r] & W_*^{\otimes n} \arrow[r] & \mathrm{Sym}^n(W_*)
\end{tikzcd}$$

so that $\mathrm{Sym}^n(f)$ is a quasi-isomorphism. Taking the direct sum, it follows that $\mathrm{Sym}(f)$ is a quasi-isomorphism as well. ☐

We can finally prove the following result:

Proposition 3.2.7. *Let k be a field of characteristic 0. Every object in $\mathbf{CAlg}_k^{\mathrm{dg}}$ has a path object.*

Proof. Let M be a direct sum of modules of the form $D^n(k)$ (cfr. Definition 3.1.23); since $0 \to D^n(k)$ is a weak equivalence and since the homology is an additive functor, it follows that $0 \to M$ is a weak equivalence as well; Lemma 3.2.6 implies then that

$$k \to \mathrm{Sym}(M)$$

is a weak equivalence in $\mathbf{CAlg}_k^{\mathrm{dg}}$ (that is, a *chain equivalence* in $\mathbf{Vect}_k^{\mathrm{dg}}$). If A is an object in $\mathbf{CAlg}_k^{\mathrm{dg}}$, then the natural map

$$A \simeq A \otimes_k k \to A \otimes_k \mathrm{Sym}(M)$$

is a weak equivalence, because $A \otimes_k -$ preserves chain homotopies.

Let A be an object in $\mathbf{CAlg}_k^{\mathrm{dg}}$ and $B := A \times A$. For each $b \in B_n$, set $C_b := D^n(k)$. Set:

$$M := \bigoplus_{b \in B} C_b$$

We have natural maps in $\mathbf{Vect}_k^{\mathrm{dg}}$

$$C_b \to B$$

which in degree n maps the generator to b and which in degree $n-1$ maps the generator to $d(b)$. These maps induce a map $M \to B$ so that we obtain, by adjunction, a map

$$\varphi\colon \mathrm{Sym}(M) \to B$$

which is surjective by construction. Moreover, we have a factorization of $\Delta\colon A \to B = A \times A$ given by

$$A \longrightarrow A \otimes_k \mathrm{Sym}(M) \xrightarrow{\Delta \otimes \varphi} B$$

The initial argument shows that $A \to A \otimes_k \mathrm{Sym}(M)$ is a weak equivalence, while $\Delta \otimes \varphi$ is degreewise surjective because φ is so. It follows that $A \otimes_k \mathrm{Sym}(M)$ is a path object for A. □

Remark 3.2.8. In the previous proof I adapted the argument given in [Hin97, Theorem 2.2.1]; observe that we could have obtained a direct proof of the factorization axiom **MC5**(i). However, the difficulty was exactly to obtain a path object; now the existence of the model structure is completely formal.

Theorem 3.2.9. *Let k be a field of characteristic 0. The category $\mathbf{CAlg}_k^{\mathrm{dg}}$ has a left proper model structure where a map $f\colon A \to B$ is a fibration or a weak equivalence if and only if $\mathcal{U}(f)$ is so.*

Proof. Every object in $\mathbf{CAlg}_k^{\mathrm{dg}}$ is fibrant and Proposition 3.2.7 guarantees that every commutative dg algebra has a path object. Proposition B.2.3 implies that a cofibration with the LLP with respect to every fibration is a weak equivalence; the hypothesis of Theorem B.2.1 are then satisfied, so that we obtain the existence of the model structure. Left properness is obvious, since every object is fibrant. □

3.3 Differential graded Lie algebras

The main goal of this section is to endow the category of differential graded Lie algebras with a good enough model structure, in order to apply the Dwyer-Kan localization and the simplicial nerve to produce an ∞-category.

3.3.1 Definition and basic constructions

Let k be a ring.

Definition 3.3.1. A differential graded Lie algebra over k is a chain complex of k-modules $(\mathfrak{g}_*, d)$ equipped with a multiplication

$$[-,-]\colon \mathfrak{g}_* \otimes_k \mathfrak{g}_* \to \mathfrak{g}_*$$

such that:

1. for homogeneous elements $x, y \in \mathfrak{g}_*$ the relation $[x,y] + (-1)^{|x||y|}[y,x] = 0$ holds;
2. for homogeneous elements $x, y, z \in \mathfrak{g}_*$ the Jacobi identity holds:

$$(-1)^{|x||z|}[x,[y,z]] + (-1)^{|x||y|}[y,[z,x]] + (-1)^{|y||z|}[z,[x,y]] = 0$$

3. the differential $d\colon \mathfrak{g}_* \to \mathfrak{g}_*$ becomes a derivation.

Definition 3.3.2. A morphism of differential graded Lie algebras over k from $(\mathfrak{g}_*, d)$ to $(\mathfrak{g}'_*, d')$ is a map of chain complexes over k, $F\colon (\mathfrak{g}_*, d) \to (\mathfrak{g}'_*, d')$, such that

$$F([x, y]) = [F(x), F(y)]$$

for every pair of homogeneous elements $x, y \in \mathfrak{g}_*$.

Notation 3.3.3. Differential graded Lie algebras over k form naturally, together with their morphisms, a category which we will denote by $\mathbf{Lie}_k^{\mathrm{dg}}$.

Example 3.3.4. Let A_* be a differential graded algebra. Define a bracket on A_* by setting

$$[x, y] := xy - (-1)^{|x||y|}yx$$

for every pair of homogeneous elements $x, y \in A_*$. Jacobi identity is easily seen to be satisfied, as well as the anticommutativity. Moreover

$$\begin{aligned} d[x, y] &= d(x)y + (-1)^{|x|}xd(y) - (-1)^{|x||y|}d(y)x - (-1)^{(|x|+1)|y|}yd(x) \\ &= [d(x), y] + (-1)^{|x|}[x, d(y)] \end{aligned}$$

It follows that this bracket makes A_* into a differential graded Lie algebra. We will denote this structure by $\mathrm{Lie}(A_*)$. It is moreover clear that in this way we defined a functor

$$\mathrm{Lie}\colon \mathbf{Alg}_k^{\mathrm{dg}} \to \mathbf{Lie}_k^{\mathrm{dg}}$$

Universal enveloping algebra

We have an obvious functor

$$\mathrm{Lie}\colon \mathbf{Alg}_k^{\mathrm{dg}} \to \mathbf{Lie}_k^{\mathrm{dg}}$$

We will construct now a left adjoint for this functor, which will be called the universal enveloping algebra functor:

$$U\colon \mathbf{Lie}_k^{\mathrm{dg}} \to \mathbf{Alg}_k^{\mathrm{dg}}$$

Consider first of all the obvious forgetful functor $\Theta\colon \mathbf{Lie}_k^{\mathrm{dg}} \to \mathbf{Mod}_k^{\mathrm{dg}}$. If $(\mathfrak{g}_*, d)$ is a differential graded Lie algebra we commit a slight abuse of notation denoting by $\mathcal{T}(\mathfrak{g}_*)$ the free (tensor) algebra associated to $\mathcal{T}(\Theta(\mathfrak{g}_*, d))$ (see Section 3.2.1). Define $U(\mathfrak{g}_*)$ to be the algebra obtained by quotienting $\mathcal{T}(\mathfrak{g}_*)$ by the two-sided ideal generated by all the expressions of the form

$$x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y]$$

where x and y range over the homogeneous elements of $\mathfrak{g}_*$. The set of such elements is stable under the differential, so that the resulting two-sided ideal will be graded. It follows that $U(\mathfrak{g}_*)$ has a natural structure of differential graded algebra.

Proposition 3.3.5. *The adjunction* $U \dashv \mathrm{Lie}$ *holds.*

Proof. Fix a differential graded Lie algebra $(\mathfrak{g}_*, d)$. We have an obvious morphism of chain complexes

$$\Theta(\mathfrak{g}_*) \to U(\mathfrak{g}_*)$$

which extends to a morphism $\mathcal{T}(\mathfrak{g}_*) \to U(\mathfrak{g}_*)$ by the universal property of the tensor algebra. Now, applying the functor Lie we obtain a morphism of differential graded Lie algebras:

$$\eta_{\mathfrak{g}_*}\colon \mathfrak{g}_* \to \mathrm{Lie}(\mathcal{T}(\mathfrak{g}_*)) \to \mathrm{Lie}(U(\mathfrak{g}_*))$$

which is easily seen to be natural in $(\mathfrak{g}_*, d)$. If now we are given a morphism of differential graded Lie algebras

$$f\colon \mathfrak{g}_* \to \mathrm{Lie}(A_*)$$

we obtain immediately a morphism $f'\colon \mathcal{T}(\mathfrak{g}_*) \to A_*$, and it is easily seen that f' sends expressions of the form

$$x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y]$$

to zero, as x and y range over the homogeneous elements of $\mathfrak{g}_*$. It follows that f' factors as map

$$\widetilde{f}\colon U(\mathfrak{g}_*) \to A_*$$

such that $\mathrm{Lie}(\widetilde{f}) \circ \eta_{\mathfrak{g}_*} = f$. Since the uniqueness of such a morphism is obvious, the proof is complete. □

It will be important to understand better the structure of the universal enveloping algebra $U(\mathfrak{g}_*)$ associated to a differential graded Lie algebra $\mathfrak{g}_*$. In the classical context, this is achieved using Poincaré-Birkhoff-Witt theorem. This theorem has an analogue in the differential graded setting, as we are going to discuss.

The universal enveloping algebra $U(\mathfrak{g}_*)$ has a filtration

$$\mathfrak{F}\colon U^{\leq 0}(\mathfrak{g}_*) \subseteq U^{\leq 1}(\mathfrak{g}_*) \subseteq U^{\leq 2}(\mathfrak{g}_*) \subseteq \cdots$$

where $U^{\leq n}(\mathfrak{g}_*)$ is the image of

$$\bigoplus_{0 \leq i \leq n} \mathfrak{g}_*^{\otimes i}$$

inside $U(\mathfrak{g}_*)$. Since the elements of the form $[x, y]$ are of degree 1 with respect to this filtration, the explicit construction of $U(\mathfrak{g}_*)$ shows that the graded algebra associated to this filtration

$$\mathrm{gr}^*_{\mathfrak{F}}(U(\mathfrak{g}_*)) := \bigoplus_{n \geq 0} U^{\leq n}(\mathfrak{g}_*)/U^{\leq n-1}(\mathfrak{g}_*)$$

is graded commutative. It follows that the natural morphism

$$\mathfrak{g}_* \to U^{\leq 1}(\mathfrak{g}_*) \to \mathrm{gr}^*_{\mathfrak{F}}(U(\mathfrak{g}_*))$$

induces a morphism

$$\Psi\colon \mathrm{Sym}(\mathfrak{g}_*) \to \mathrm{gr}^*_{\mathfrak{F}}(U(\mathfrak{g}_*))$$

Theorem 3.3.6 (Poincaré-Birkhoff-Witt)**.** *The morphism Ψ is an isomorphism of differential graded commutative algebras.*

Proof. See [Qui69, Theorem B.2.3] for an equivalent statement and a proof. □

Corollary 3.3.7. *Let k be a field of characteristic zero and let $\mathfrak{g}_*$ be a differential graded Lie algebra. There is an isomorphism of chain complexes between* $\mathrm{Sym}(\mathfrak{g}_*)$ *and* $U(\mathfrak{g}_*)$*.*

Proof. Let $\psi_n\colon \mathfrak{g}_*^{\otimes n} \to U(\mathfrak{g}_*)$ be the multiplication map. For every permutation on n elements σ, let ϕ_σ denote the induced automorphism of $\mathfrak{g}_*^{\otimes n}$. The induced map

$$\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \psi \circ \phi_\sigma$$

is invariant under precomposition with each of the maps ϕ_σ, so that it factors as

$$\begin{tikzcd}
\mathfrak{g}_*^{\otimes n} \arrow[r, "\psi_n"] \arrow[d] & U(\mathfrak{g}_*) \\
\mathrm{Sym}^n(\mathfrak{g}_*) \arrow[r, "\Psi_n"] & U(\mathfrak{g}_*)^{\leq n} \arrow[u, hook]
\end{tikzcd}$$

We obtain in this way a morphism of *chain complexes*

$$\Psi\colon \mathrm{Sym}^*(\mathfrak{g}_*) \to U(\mathfrak{g}_*)$$

Since the composite

$$\mathrm{Sym}^n(\mathfrak{g}_*) \xrightarrow{\psi_n} U(\mathfrak{g}_*)^{\leq n} \longrightarrow \mathrm{gr}^n_{\mathfrak{F}}(U(\mathfrak{g}_*))$$

is an isomorphism of chain complexes in virtue of Poincaré-Birkhoff-Witt theorem, it follows that

$$\mathrm{gr}^*_{\mathfrak{F}}(\Psi)\colon \mathrm{gr}^*_{\mathfrak{F}}(\mathrm{Sym}^*(\mathfrak{g}_*)) \to \mathrm{gr}^*_{\mathfrak{F}}(U(\mathfrak{g}_*))$$

is an isomorphism, so that Ψ is an isomorphism as well. □

Free DGLA construction

We now use the universal enveloping algebra to construct explicitly a free Lie algebra functor. We have an obvious forgetful functor

$$\Theta\colon \mathbf{Lie}_k^{\mathrm{dg}} \to \mathbf{Mod}_k^{\mathrm{dg}}$$

This functor has a left adjoint, which we will denote by

$$\mathfrak{f}\colon \mathbf{Mod}_k^{\mathrm{dg}} \to \mathbf{Lie}_k^{\mathrm{dg}}$$

We remark that the existence of this functor can be obtained in a purely formal way, using the adjoint functor theorem. To obtain a direct construction, we can proceed as follows: if M_* is a chain complex of k-modules, we consider the free algebra $\mathcal{T}(M_*)$ introduced in section 3.2.1; example 3.3.4 shows that $\mathrm{Lie}(\mathcal{T}(M_*))$ has a structure of differential graded Lie algebra. Define $\mathfrak{f}(M_*)$ to be the Lie subalgebra of $\mathrm{Lie}(\mathcal{T}(M_*))$ spanned by the image of M_* in $\mathrm{Lie}(\mathcal{T}(M_*))$.

Proposition 3.3.8. *The adjunction relation* $\mathfrak{f} \dashv \Theta$ *holds.*

Proof. We have a morphism of chain complexes

$$\eta_{M_*}\colon M_* \to \Theta(\mathfrak{f}(M_*))$$

which is easily seen to be natural in M_*. If we are given a morphism of chain complexes $f\colon M_* \to \Theta(\mathfrak{g}_*)$ we define obtain a unique functor $\mathcal{T}(M_*) \to U(\mathfrak{g}_*)$; applying Lie we get a morphism

$$\widetilde{f}\colon \mathrm{Lie}(\mathcal{T}(M_*)) \to \mathrm{Lie}(U(\mathfrak{g}_*)) \to \mathfrak{g}_*$$

which is such that $\Theta(\widetilde{f}) \circ \eta_{M_*} = f$. Since the uniqueness of such a functor is clear, we obtain the thesis. □

3.3.2 Model structure

Let k be a field of characteristic 0. We would like to apply Theorem B.2.1 to the adjunction

$$\mathfrak{f}\colon \mathbf{Vect}_k^{\mathrm{dg}} \rightleftarrows \mathbf{Lie}_k^{\mathrm{dg}} : \Theta$$

Define a morphism $f\colon \mathfrak{g}_* \to \mathfrak{h}_*$ in $\mathbf{Lie}_k^{\mathrm{dg}}$ to be a weak equivalence or a fibration if and only if $\Theta(f)$ is so. We immediately have the following result:

Lemma 3.3.9. *A morphism of differential graded Lie algebras* $f\colon \mathfrak{g}_* \to \mathfrak{h}_*$ *is a weak equivalence if and only if* $U(f)\colon U(\mathfrak{g}_*) \to U(\mathfrak{h}_*)$ *is a weak equivalence of differential graded algebras.*

Proof. This is an immediate consequence of Lemma 3.2.6 and Corollary 3.3.7. □

Theorem 3.3.10. *Let k be a field of characteristic* 0*. Then the weak equivalences and the fibrations we defined endow* $\mathbf{Lie}_k^{\mathrm{dg}}$ *with a left proper and combinatorial model structure.*

Proof. A fully detailed proof can be found in [DAGX, Proposition 2.1.10]. Another argument is possible using the transfer principle of Theorem B.2.1. □

Corollary 3.3.11. *The adjoint pair*

$$\mathfrak{f}\colon \mathbf{Vect}_k^{\mathrm{dg}} \rightleftarrows \mathbf{Lie}_k^{\mathrm{dg}} : \Theta$$

is a Quillen pair.

Proof. We simply observe that the functor Θ preserves fibrations and weak equivalences by definition. □

Corollary 3.3.12. *The adjoint pair*

$$U\colon \mathbf{Lie}_k^{\mathrm{dg}} \rightleftarrows \mathbf{Alg}_k^{\mathrm{dg}} \colon \mathrm{Lie}$$

is a Quillen pair.

Proof. By definition, the functor Lie preserves both fibrations and weak equivalences. □

We will denote by Lie_k the ∞-category underlying $\mathbf{Lie}_k^{\mathrm{dg}}$. Observe that the forgetful functor

$$\Theta\colon \mathbf{Lie}_k^{\mathrm{dg}} \to \mathbf{Vect}_k^{\mathrm{dg}}$$

preserves weak equivalences, so that it induces a forgetful functor

$$\theta\colon \mathrm{Lie}_k \to \mathrm{Mod}_k$$

where Mod_k is the ∞-category underlying $\mathbf{Vect}_k^{\mathrm{dg}}$.

Lemma 3.3.13. *The ∞-category* Lie_k *is presentable and the forgetful functor θ preserves small sifted colimits.*

Proof. The first statement is the content of [HA, Proposition 1.3.3.9]. The remaining argument can be found in [DAGX, Proposition 2.1.16]. □

3.3.3 Chevalley-Eilenberg complexes of $\mathfrak{g}_*$

If $\mathfrak{g}$ is a Lie algebra, it is possible to define a notion of (right) $\mathfrak{g}$-module; the category of such objects turn out to be equivalent to the category of $U(\mathfrak{g})$-modules, making clear that it has enough projectives and injectives. Adopting this point of view, one defines then the homology and cohomology groups of $\mathfrak{g}$ as

$$H_n(\mathfrak{g}) := \mathrm{Tor}_n^{U(\mathfrak{g})}(k,k), \qquad H^n(\mathfrak{g}) := \mathrm{Ext}^n_{U(\mathfrak{g})}(k,k)$$

The Chevalley-Eilenberg groups of $\mathfrak{g}$ are classically used to compute these groups in an explicit way. In this last part, we develop a similar theory in the differential graded setting.

Homological Chevalley-Eilenberg complex

Let $\mathfrak{g}_*$ be a fixed differential graded Lie algebra. Consider the cone of $\mathfrak{g}_*$ as a chain complex:

$$\mathrm{Cone}(\mathfrak{g}_*) := \mathrm{Cone}(\Theta(\mathfrak{g}_*))$$

We will denote an element in $\mathrm{Cone}(\mathfrak{g}_*)_n$ as $x+\varepsilon y$. Using this notation, the usual differential in $\mathrm{Cone}(\mathfrak{g}_*)$ becomes

$$d(x+\varepsilon y) := dx + y - \varepsilon dy \tag{3.2}$$

We define a bracket on $\mathrm{Cone}(\mathfrak{g}_*)$ by setting

$$[x+\varepsilon y, x' + \varepsilon y'] := [x,x'] + \varepsilon([y,x'] + (-1)^{|x|}[x,y'])$$

as x ranges over the homogeneous elements of $\mathfrak{g}_*$.

Lemma 3.3.14. *With the bracket defined above,* $\mathrm{Cone}(\mathfrak{g}_*)$ *is a differential graded Lie algebra.*

Proof. This is a straightforward verification. □

Definition 3.3.15. Let $\mathfrak{g}_*$ be a differential graded Lie algebra. We define the homological Chevalley-Eilenberg complex of $\mathfrak{g}_*$ to be

$$C_*(\mathfrak{g}_*) := U(\mathrm{Cone}(\mathfrak{g}_*)) \otimes_{U(\mathfrak{g}_*)} k$$

In order to better understand the underlying chain complex of $C_*(\mathfrak{g}_*)$, let us consider the categories $\mathbf{GLA}_k$ of *graded Lie algebras over k* and $\mathbf{CGA}_k$ of *graded commutative algebras over k* (without differential). We have an obvious forgetful functor

$$F\colon \mathbf{CGA}_k \to \mathbf{GLA}_k$$

which operates as the functor Lie of example 3.3.4. The same construction we gave in 3.3.1 produces a left adjoint to F, which we still denote

$$U\colon \mathbf{GLA}_k \to \mathbf{CGA}_k$$

Observe that the forgetful functor $\mathbf{GLA}_k \to \mathbf{GMod}_k$ creates coproducts. We therefore see that $\mathrm{Cone}(\mathfrak{g}_*)$ is the coproduct of $\mathfrak{g}_*$ and $\mathfrak{g}_*[-1]$ in the category $\mathbf{GLA}_k$, where the last one has vanishing bracket. Applying the universal enveloping functor we get an isomorphism of graded algebras:

$$U(\mathrm{Cone}(\mathfrak{g}_*)) \simeq U(\mathfrak{g}_*[-1]) \otimes_k U(\mathfrak{g}_*)$$

because U commutes with coproducts being a left adjoint. Using the analogue of Poincaré-Birkhoff-Witt theorem for these categories we deduce that there is an isomorphism of chain complexes

$$\mathrm{Sym}^*(\mathfrak{g}_*[-1]) \simeq U(\mathfrak{g}_*[-1])$$

Therefore we obtain a nice description of the underlying chain complex of $C_*(\mathfrak{g}_*)$:

$$C_*(\mathfrak{g}_*) \simeq \mathrm{Sym}^*(\mathfrak{g}_*[-1]) \tag{3.3}$$

The canonical filtration on $\mathrm{Sym}^*(\mathfrak{g}_*[-1])$ induces, via this identification, a filtration on $C_*(\mathfrak{g}_*)$:

$$k \simeq C_*^{\leq 0}(\mathfrak{g}_*) \subseteq C_*^{\leq 1}(\mathfrak{g}_*) \subseteq C_*^{\leq 2}(\mathfrak{g}_*) \subseteq \cdots$$

Lemma 3.3.16. *There is an isomorphism in the category of chain complexes*

$$C_*^{\leq n}(\mathfrak{g}_*)/C_*^{\leq n-1}(\mathfrak{g}_*) \simeq \mathrm{Sym}^n(\mathfrak{g}_*)$$

Proof. We claim that under the identification (3.3), the differential on $C_*(\mathfrak{g}_*)$ is given by the formula

$$\begin{aligned} D(x_1 \cdots x_n) = & \sum_{1 \leq i \leq n} (-1)^{p_1 + \ldots + p_{i-1}} x_1 \cdots x_{i-1} dx_i x_{i+1} \cdots x_n + \\ & \sum_{1 \leq i < j \leq n} \eta x_1 \cdots x_{i-1} x_{i+1} \cdots x_{j-1} [x_i, x_j] x_{j+1} \cdots x_n \end{aligned}$$

where $\eta \in \{\pm 1\}$ is a sign. Observe that, assuming this statement, the lemma is completely obvious. As for the computation, start by considering the natural map of *graded Lie algebras* $\mathfrak{g}_*[-1] \to \mathrm{Cone}(\mathfrak{g}_*)$. The image of an element $x_1 \cdots x_n \in \mathrm{Sym}^*(\mathfrak{g}_*[-1])$ inside $C_*(\mathfrak{g}_*)$ is obtained as the image of the element

$$\varepsilon x_1 \otimes \ldots \otimes \varepsilon x_n \in U(\mathfrak{g}_*[-1])$$

via the map (of graded algebras)

$$U(\mathfrak{g}_*[-1]) \to U(\mathfrak{g}_*[-1]) \otimes_k U(\mathfrak{g}_*) \to U(\mathfrak{g}_*[-1]) \otimes_k U(\mathfrak{g}_*) \otimes_{U(\mathfrak{g}_*)} k \simeq U(\mathfrak{g}_*[-1])$$

The differential coming from the definition of $C_*(\mathfrak{g}_*)$ is the one induced from the differential (3.2) of the cone, so that computing shows that $D(x_1 \cdots x_n)$ correspond under the above identification to

$$\begin{aligned} & \sum_{i=1}^n (-1)^{q(1,i)} \varepsilon x_1 \otimes \cdots \otimes \varepsilon x_{i-1} \otimes (x_i - \varepsilon d x_i) \otimes \ldots \otimes \varepsilon x_n = \\ & \quad = \sum_{i=1}^n (-1)^{q(1,i)} \varepsilon x_1 \otimes (-dx_i) \otimes \ldots \otimes \varepsilon x_n + \sum_{i=1}^n (-1)^{q(1,i)} \varepsilon x_1 \otimes \ldots \otimes x_i \otimes \ldots \otimes \varepsilon x_n \end{aligned}$$

where $q(i,j) = p_i + \ldots + p_{j-1}$. Now, under the identification (3.3) the element

$$\varepsilon x_1 \otimes (-dx_i) \otimes \ldots \otimes \varepsilon x_n$$

corresponds to $x_1 \cdots dx_i \cdots x_n$ (because the differential on $\mathfrak{g}_*[-1]$ is changed of sign). On the other side, using the equation given by the ideal defining $U(\mathrm{Cone}(\mathfrak{g}_*))$ we can rewrite

$$\begin{aligned}(-1)^{q(1,i)}\varepsilon x_1 \otimes \ldots \otimes x_i \otimes \varepsilon x_{i+1} \otimes \ldots \otimes \varepsilon x_n =&(-1)^{q(1,i+2)}\varepsilon x_1 \otimes \ldots \varepsilon x_{i+1} \otimes x_i \otimes \ldots \otimes \varepsilon x_n + \\ &+ (-1)^{q(1,i)}\varepsilon x_1 \otimes \ldots \otimes [x_i, \varepsilon x_{i+1}] \otimes \ldots \otimes \varepsilon x_n\end{aligned}$$

Observing that $[x_i, \varepsilon x_{i+1}] = (-1)^{p_i}\varepsilon[x_i, x_{i+1}]$ and iterating this process, we eventually arrive to the form

$$\sum_{i<j} \eta_j \varepsilon x_1 \otimes \ldots \otimes \varepsilon[x_i, x_j] \otimes \ldots \otimes \varepsilon x_n + \eta_n \varepsilon x_1 \otimes \ldots \otimes \varepsilon x_n \otimes x_i$$

and the last element is zero in the identification (3.3). The claim is therefore completely proved. $\square$

Proposition 3.3.17. *Let k be a field of characteristic 0; let $f\colon \mathfrak{g}_* \to \mathfrak{h}_*$ be a quasi-isomorphism between differential graded Lie algebras. The induced map*

$$C_*(f)\colon C_*(\mathfrak{g}_*) \to C_*(\mathfrak{h}_*)$$

is a quasi-isomorphism of chain complexes.

Proof. Using the filtration on $C_*(\mathfrak{g}_*)$ introduced above, one is immediately reduced to show that the induced maps $\theta_n\colon C_*^{\leq n}(\mathfrak{g}_*) \to C_*^{\leq n}(\mathfrak{h}_*)$ are quasi-isomorphism. This can be dealt with by induction on n: if $n = 0$, this is the hypothesis. Otherwise, previous lemma shows that there is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & C_*^{\leq n-1}(\mathfrak{g}_*) \arrow[r] \arrow[d, "\theta_{n-1}"] & C_*^{\leq n}(\mathfrak{g}_*) \arrow[r] \arrow[d, "\theta_n"] & \mathrm{Sym}^n(\mathfrak{g}_*[-1]) \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & C_*^{\leq n-1}(\mathfrak{h}_*) \arrow[r] & C_*^{\leq n}(\mathfrak{h}_*) \arrow[r] & \mathrm{Sym}^n(\mathfrak{h}_*[-1]) \arrow[r] & 0 \end{tikzcd}$$

The inductive hypothesis shows that θ_{n-1} is an isomorphism, while the proof of Lemma 3.2.6 implies that ϕ is a quasi-isomorphism. It follows that θ_n is a quasi-isomorphism as well, completing the proof. $\square$

As consequence, we see that the functor

$$C_*\colon \mathbf{Lie}_k^{\mathrm{dg}} \to \mathbf{Vect}_k^{\mathrm{dg}}$$

preserve quasi-isomorphism. Therefore, it induces a functor of ∞-categories

$$C_*\colon \mathrm{Lie}_k \to \mathrm{Mod}_k$$

Proposition 3.3.18. *Let k be a field of characteristic zero. The functor of ∞-categories*

$$C_*\colon \mathrm{Lie}_k \to (\mathrm{Mod}_k)_{k/}$$

preserves small colimits.

Proof. See [DAGX, Proposition 2.2.12]. $\square$

Example 3.3.19. Let V_* be a chain complex over a field k of characteristic zero. Let $\mathfrak{g}_* := \mathfrak{f}(V_*)$ be the free Lie algebra generated by V_*. Then the homological Chevalley-Eilenberg of $\mathfrak{g}_*$ has a quasi-isomorphic description given by

$$\xi\colon k \oplus V_*[-1] \to k \oplus \mathfrak{g}_*[-1] \simeq C_*^{\leq 1}(\mathfrak{g}_*) \subset C_*(\mathfrak{g}_*)$$

The verification that ξ is a quasi-isomorphism can be tricky. We refer to [DAGX, Proposition 2.2.7] for a complete proof.

Cohomological Chevalley-Eilenberg complex

We now introduce the cohomological Chevalley-Eilenberg complex. Let $\mathfrak{g}_*$ be a differential graded Lie algebra. Using example 3.1.21 we can identify

$$\mathbf{Hom}_k^{\mathrm{dg}}(C_*(\mathfrak{g}_*), k) \simeq \mathrm{Hom}_k(C_*(\mathfrak{g}_*), k)$$

We set

$$C^*(\mathfrak{g}_*) := \mathrm{Hom}_k(C_*(\mathfrak{g}_*), k)$$

and we will identify elements $\lambda \in C^n(\mathfrak{g}_*)$ with the dual space of $\mathrm{Sym}^n(\mathfrak{g}_*[-1])$.

We can endow C^* with a cup-product. Namely, if $\lambda \in C^p(\mathfrak{g}_*)$ and $\mu \in C^q(\mathfrak{g}_*)$ we define $\lambda\mu \in C^{p+q}(\mathfrak{g}_*)$ via the formula

$$(\lambda\mu)(x_1 \cdots x_n) := \sum_{S,S'} \varepsilon(S,S')\lambda(x_{i_1} \cdots x_{i_m})\mu(x_{j_1} \cdots x_{j_{n-m}})$$

where the x_i are homogeneous elements of $\mathfrak{g}_*$ and the sum is taken over all disjoint sets $S = \{i_1 < \ldots < i_m\}$ and $S' = \{j_1 < \ldots < j_{n-m}\}$ such that $S \cup S' = \{1, \ldots, n\}$ and

$$|x_{i_1}| + \ldots + |x_{i_m}| = p$$

$\varepsilon(S,S')$ is defined by the formula

$$\varepsilon(S,S') := \prod_{i \in S', j \in S, i<j} (-1)^{|x_i||x_j|}$$

Lemma 3.3.20. *With the multiplication we defined above ,$C^*(\mathfrak{g}_*)$ becomes a commutative differential graded algebra.*

Proof. This is a simple verification. □

Observe that $C^*(0) = k$. The natural map $0 \to \mathfrak{g}_*$ induces therefore an augmentation

$$C^*(\mathfrak{g}_*) \to k$$

which implies that C^* defines a functor

$$C^* \colon \mathbf{Lie}_k^{\mathrm{dg}} \to (\mathbf{CAlg}_k^{\mathrm{op}})_{k/} \simeq (\mathbf{CAlg}_k^{\mathrm{aug}})^{\mathrm{op}}$$

Since we are working over a field, the additivity of $\mathrm{Hom}_k(-, k)$ implies that this functor preserves quasi-isomorphisms. It follows immediately from Proposition 3.3.17 that also C^* preserves quasi-isomorphisms. Therefore, it induces a functor of ∞-categories:

$$C^* \colon \mathrm{Lie}_k \to (\mathrm{CAlg}_k^{\mathrm{aug}})^{\mathrm{op}}$$

Example 3.3.21. let V_* be a chain complex over a field of characteristic zero. The quasi-isomorphism of example 3.3.19 induces a quasi isomorphism

$$C^*(\mathfrak{g}_*) \to k \oplus V_*^\vee[1]$$

Endowing $k \oplus V_*^\vee[1]$ with the structure of a trivial square-zero extension, we see that the morphism $C^*(\mathfrak{g}_*) \to k \oplus V_*^\vee[1]$ is a quasi-isomorphism of commutative differential graded algebras.

Proposition 3.3.22. *Let k be a field of characteristic zero. The functor $C^* \colon \mathrm{Lie}_k \to (\mathbf{CAlg}_k^{\mathrm{aug}})^{\mathrm{op}}$ preserves small colimits.*

Proof. See [DAGX, Proposition 2.2.17]. □

Chapter 4

Formal moduli problems

In this final chapter we adapt the general theory developed in [DAGX] to the case of commutative differential graded algebras. The main result can be described as follows: suppose we are given a derived stack $X \colon \mathfrak{Y} \to \mathcal{S}$, where $\mathfrak{Y}$ is a presentable ∞-category; if we fix a point $\eta \in X(*)$, we would like to be able to describe the formal neighbourhood of X at the point η; reasoning as in Chapter 1, we see that this is equivalent to restrict the stack X to a well-chosen full subcategory $\mathfrak{Y}^{\mathrm{sm}} \subset \mathfrak{Y}$ of *small objects*. Such restrictions are called *formal moduli problems*; a first important idea is that it should be possible to study the "tangent complex" T_X of such a functor, and that such tangent space should give many informations about the problem itself: this is formalized in Theorem 4.2.10.

Next, one could observe that the operation of taking the tangent complex

$$X \mapsto T_X$$

commutes with finite limits. It follows then that $T(\Omega X) \simeq \Omega T_X \simeq T_X[1]$. Since ΩX is to be thought as a group object in the category of formal moduli problems, we would expect a Lie algebra structure in $T_X[1]$. This is a piece of the main theorem of this chapter; the second part, and probably the most interesting one, is that this Lie algebra structure completely determines the formal moduli problem X. We will express this by saying that there is an equivalence of ∞-categories between the category of formal moduli problems and the category Lie_k introduced in the previous chapter.

4.1 Formal deformation theory

An elegant exposition of the main topics of formal deformation theory can be found in [HA, Chapter 8.4]. In this preliminary section we recall the main ideas and state the main theorems, without going into the technical details of the proofs. We will need such results later, in order to develop a consistent theory of formal moduli problems.

4.1.1 Derivations

Let k be a field of characteristic zero. Let $\mathbf{CAlg}_k$ be the category of differential graded commutative k-algebras, endowed with the monoidal model structure of previous chapter. Let $\mathbf{Vect}_k^{\mathrm{dg}}$ be the category of complexes of k-modules. Given $A \in \mathrm{Ob}(\mathbf{CAlg}_k)$ and $M \in \mathrm{Ob}(\mathbf{Mod}_A)$ we can define a new element of $\mathbf{CAlg}_k$ in the following way: endow the complex $A \oplus M$ with the algebra structure given by

$$(x, m) \cdot (y, n) := (xy, xn + (-1)^{|x||y|} ym)$$

This is the analogue for $\mathbf{CAlg}_k$ of the square-zero extension defined in Chapter 1. The construction being obviously functorial, we obtain a functor

$$F_A \colon \mathbf{Mod}_A \to \mathbf{CAlg}_A$$

In particular, we obtain $F = F_k \colon \mathbf{Vect}_k^{\mathrm{dg}} \to \mathbf{CAlg}_k$. Observe that if M is a complex of k-modules, then $F(M) = k \oplus M$ is equipped with a natural projection

$$k \oplus M \to k$$

which makes $F(M)$ an object of $(\mathbf{CAlg}_k)_{/k}$.

Using the canonical equivalence, in the classical context, between derivations and sections of trivial square-zero extensions, we introduce the following definition (cfr. also [TV08, Definition 1.2.1.1])

Definition 4.1.1. Let A be a commutative differential graded k-algebra and let M be a differential graded k-module. A k-derivation of A into M is an element of

$$\mathrm{Hom}_k(A, F(M))$$

We will denote the set of derivations by $\mathrm{Der}_k(A, M)$.

Remark 4.1.2. It is possible to build a more refined version of $\mathrm{Der}_k(A, M)$, endowing it with the structure of ∞-category. We won't pursue this point of view in this mémoire, but the interested reader can find it in [HA, Chapter 8.4].

The cotangent complex can be introduced in this context as well as we did in the first chapter. For more details on the construction, we will refer to [TV08, Section 1.2.1] and [HA, Section 8.3].

Theorem 4.1.3. *Let $f \colon A \to B$ be a morphism between connective commutative differential graded algebras. If $\mathrm{cofib}(f)$ is n-connective for some $n \geq 0$, then there is a canonical $(2n)$-connective map of B-modules $B \otimes_A \mathrm{cofib}(f) \to \mathbb{L}_{B/A}$.*

Proof. See [HA, Theorem 8.4.3.1]. □

4.1.2 Small morphisms

Definition 4.1.4. A morphism $\phi \colon A' \to A$ in $\mathbf{CAlg}_k$ is *elementary* if there exists an integer $n > 0$ and a homotopy pullback diagram

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "\phi"'] & k \arrow[d] \\ A \arrow[r] & k \oplus k[n] \end{tikzcd}$$

Lemma 4.1.5. *A morphism $\phi \colon A' \to A$ in $\mathbf{CAlg}_k$ is elementary if and only if it satisfies the following conditions:*

1. $\mathrm{fib}(\phi) \simeq k[n]$ *for some* $n \geq 0$ *in* A'-$\mathbf{Mod}$*;*
2. *if* $n = 0$*, the map* $\pi_0\mathrm{fib}(\phi) \otimes_{\pi_0 A'} \pi_0\mathrm{fib}(\phi) \to \pi_0\mathrm{fib}(\phi)$ *vanishes.*

Proof. Assume first that ϕ is elementary. Proposition 4.1.9 implies that ϕ is equivalent to a map $\psi \colon A'' \to A$ fitting in a pullback diagram as follows:

$$\begin{tikzcd} A'' \arrow[r] \arrow[d, "\psi"'] & k \oplus \mathrm{Cone}(k[n-1]) \arrow[d] \\ A \arrow[r] & k \oplus k[n] \end{tikzcd}$$

In particular $\mathrm{fib}(\phi) \simeq \mathrm{fib}(\psi)$. Since ψ is surjective, it follows that ψ is a fibration in A'-$\mathbf{Mod}$; the coglueing lemma implies immediately that $\ker\psi$ is an explicit model for $\mathrm{fib}(\psi)$ in A'-$\mathbf{Mod}$. Standard homological algebra shows then that

$$\begin{aligned} \ker\psi &\simeq \ker(k \oplus \mathrm{Cone}(k[n-1]) \to k \oplus k[n]) \\ &\simeq \ker(\mathrm{Cone}(k[n-1]) \to k[n]) \simeq k[n-1] \end{aligned}$$

If moreover $\ker\psi \simeq k[0]$ and $x \in \pi_0 \ker\psi$ is a generator, then the algebra structure on $k \oplus \mathrm{Cone}(k[1])$ implies that $x^2 = 0$, so that the thesis follows. Conversely, one can use the equivalence of [HA, Theorem 8.4.1.26] in order to conclude. □

Definition 4.1.6. A morphism $\phi\colon A' \to A$ in $(\mathbf{CAlg}_k)_{/k}$ is said to be *small* if it can be written as a composition of finitely many elementary morphisms. An object A in $(\mathbf{CAlg}_k)_{/k}$ is *small* if the map $A \to k$ is small.

The notion of small object should be thought as a generalization in the derived setting of the notion of artinian rings. To make precise this statement, consider the following result:

Proposition 4.1.7. *An object A in $(\mathbf{CAlg}_k)_{/k}$ is small if and only if the following conditions are satisfied:*

1. *the homotopy groups $\pi_n A$ vanish for $n < 0$ and $n \gg 0$;*
2. *each $\pi_n A$ is finite dimensional over k;*
3. *$\pi_0 A$ is local with maximal ideal $\mathfrak{m}$ and $k \to \pi_0 A/\mathfrak{m}$ is an isomorphism.*

Proof. Assume that A is small and factorize the map $A \to k$ as

$$A = A_0 \to A_1 \to \cdots \to A_n \simeq k$$

where $A_i \to A_{i+1}$ is a square-zero extension of A_{i+1} by $k[n_i]$, for some integer $n_i \geq 0$. We prove that A_i satisfies the conditions *1.* to *3.* by descending induction on i, the case $i = n$ being obvious. If the statement holds for $i+1$, consider the fiber sequence whose existence is guaranteed by Lemma 4.1.5:

$$k[n_i] \to A_i \to A_{i+1}$$

The long exact sequence of homotopy groups associated to this fiber sequence implies

$$\pi_n A_i = 0$$

for $n < 0$ and $n \gg 0$. Finally, being $A_i \to A_{i+1}$ a square-zero extension, we see that $\ker(\pi_0\phi)$ is a square-zero ideal of $\pi_0 A_i$, so that we have an extension

$$0 \to \ker(\pi_0\phi) \to \pi_0 A_i \to \pi_0 A_{i+1} \to 0$$

showing that $\pi_0 A_i$ is local artinian as well.

Conversely, assume that A satisfies conditions *1.* to *3.* We proceed by induction on $\dim_k \pi_* A$. Let n be the largest integer such that $\pi_n A \neq 0$; if $n = 0$, then $A \simeq \pi_0 A$ is discrete and condition *3.* implies that it is an artinian ring with maximal ideal $\mathfrak{m}$. In particular $\mathfrak{m}^i = 0$ for some integer $i \geq 0$. Let i be the minimal integer with this property; if $i = 0$, then $A \simeq k$ and we are done. Otherwise, choose a nonzero element $x \in \mathfrak{m}^{i-1}$; by construction $x^2 = 0$, so that

$$0 \to (x) \to A \to A/(x) \to 0$$

is a square-zero extension. In particular, Lemma 4.1.5 implies that $A \to A/(x)$ is elementary. Since $A/(x)$ is small by hypothesis, we are done.

Suppose now $n > 0$ and set $M = \pi_n A$. Let $\mathfrak{m} \subset \pi_0 A$. Since $\pi_0 A$ is artinian, the condition $\mathfrak{m}^i M = 0$ holds for $i \gg 0$; choose the minimal integer i such that $\mathfrak{m}^{i+1} M = 0$ and let $x \in \mathfrak{m}^i M$ be a nonzero element; write $M' = M/xM$. If $A'' := \tau_{\leq n-1} A$ denotes the Postnikov truncation of A, we obtain via [HA, Theorem A.8.4.1.26] (see especially Remark 8.4.1.29 there) a homotopy pullback diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & k \arrow[d] \\ A'' \arrow[r] & k \oplus M[n+1] \end{tikzcd}$$

Set

$$A' := A'' \times_{k \oplus M'[n+1]} k$$

Then $A \simeq A' \times_{k \oplus k[n+1]} k$, so that the map $A \to A'$ is elementary. Since A' is small by induction, if follows that A is small. $\square$

4.1.3 A spectrum object

Let
$$F\colon \mathbf{Vect}_k^{\mathrm{dg}} \to (\mathbf{CAlg}_k)_{/k}$$
be the square-zero extension functor introduced at the beginning of this section. It has the following pleasant property:

Lemma 4.1.8. *The functor $F\colon \mathbf{Vect}_k^{\mathrm{dg}} \to (\mathbf{CAlg}_k)_{/k}$ preserves pullbacks.*

Proof. Let
$$\begin{tikzcd} M_{01} \arrow[r, "p"] \arrow[d, "q"'] & M_0 \arrow[d, "f"] \\ M_1 \arrow[r, "g"'] & N \end{tikzcd}$$
be a pullback square. We want to show that the induced diagram is a pullback in $(\mathbf{CAlg}_k)_{/k}$. Let $A \to k$ be a commutative dg k-algebra that fits into the following diagram
$$\begin{tikzcd} A \arrow[dr, dotted, "\alpha"] \arrow[drr, bend left, "\alpha_0"] \arrow[ddr, bend right, "\alpha_1"'] & & \\ & k \oplus M_{01} \arrow[r] \arrow[d] & k \oplus M_0 \arrow[d, "1\oplus f"] \\ & k \oplus M_1 \arrow[r, "1\oplus g"'] & k \oplus N \end{tikzcd}$$
Using Lemma **??**, let $d_0\colon A \to M_0$ and $d_1\colon A \to M_1$ be the derivations corresponding to α_0 and α_1 respectively. Since the square is commutative, we obtain
$$f \circ d_0 = g \circ d_1$$
In particular, we obtain a map $d\colon A \to M_{01}$. The universal property of the pullback allows to verify immediately that d is a derivation; its corresponding map
$$\alpha\colon A \to k \oplus M_{01}$$
is the (unique) map we were looking for. □

Proposition 4.1.9. *Let $n > 0$ be an integer. In $(\mathbf{CAlg}_k)_{/k}$ the natural map $k \to k \oplus k[n]$ has a factorization*
$$k \xrightarrow{i} k \oplus \mathrm{Cone}(k[n+1]) \xrightarrow{p} k \oplus k[n]$$
where i is a weak equivalence and p is a fibration.

Proof. The map
$$0 \to \mathrm{Cone}(k[n+1])$$
is a chain equivalence in $\mathbf{Vect}_k^{\mathrm{dg}}$. It follows that the map
$$k \to k \oplus \mathrm{Cone}(k[n+1])$$
is a chain equivalence as well. Moreover, we have a natural map
$$\mathrm{Cone}(k[n+1]) \to k[n]$$
which induces a fibration
$$k \oplus \mathrm{Cone}(k[n+1]) \to k \oplus k[n]$$
Considering $k \oplus \mathrm{Cone}(k[n+1]) = F(\mathrm{Cone}(k[n+1]))$ as an element in $(\mathbf{CAlg}_k)_{/k}$, we obtain the desired factorization of $k \to k \oplus k[n]$ as
$$k \longrightarrow k \oplus \mathrm{Cone}(k[n+1]) \longrightarrow k \oplus k[n]$$
The definitions of fibrations and weak equivalences in $(\mathbf{CAlg}_k)_{/k}$ imply the thesis. □

Corollary 4.1.10. *In $(\mathbf{CAlg}_k)_{/k}$, there is a natural isomorphism $\Omega(k \oplus k[n]) \simeq k \oplus k[n+1]$.*

Proof. By definition, $\Omega(k \oplus k[n])$ is the homotopy pullback of

$$k \longrightarrow k \oplus k[n] \longleftarrow k$$

Since in $(\mathbf{CAlg}_k)_k$ every object is fibrant, the coglueing lemma jointly with Proposition 4.1.9 implies that an explicit model for $\Omega(k \oplus k[n])$ is given by the pullback of

$$k \oplus \mathrm{Cone}(k[n+1]) \xrightarrow{p} k \oplus k[n] \longleftarrow k$$

is a model for $\Omega(k \oplus k[n])$. Since

$$\begin{tikzcd} k[n-1] \arrow[r] \arrow[d] & 0 \arrow[d] \\ \mathrm{Cone}(k[n+1]) \arrow[r] & k[n] \end{tikzcd}$$

is a pullback diagram in $\mathbf{Vect}_k^{\mathrm{dg}}$, Lemma 4.1.8 implies immediately that

$$\begin{tikzcd} k \oplus k[n+1] \arrow[r] \arrow[d] & k \arrow[d] \\ k \oplus \mathrm{Cone}(k[n+1]) \arrow[r] & k \oplus k[n] \end{tikzcd}$$

is a pullback in $(\mathbf{CAlg}_k)_{/k}$. □

We immediately obtain the following result:

Corollary 4.1.11. *The sequence $\{k \oplus k[n]\}_{n \geq 0}$ is a spectrum in $(\mathbf{CAlg}_k)_{/k}$.*

4.2 Formal moduli problems

In this section we analyze the notion of formal moduli problems. A formal moduli problem should be thought of as a derived analogue of a deformation functor, in the sense introduced in Chapter 1. After introducing the definition, we explore the idea of tangent space to a formal moduli problem and we show that it can be obtained as infinite loop space associated to a spectrum object. Finally, we introduce the notion of smoothness for a map between formal moduli problems; we conclude the section showing that every formal moduli problem has a presentation given by a smooth hypercover.

4.2.1 Definition

Notation 4.2.1. We will denote by $\mathcal{S}$ the ∞-category of spaces, that is the ∞-category associated to **sSet** with the standard model structure.

We will denote by $\mathbf{CAlg}_k^{\mathrm{sm}}$ the full subcategory of the ∞-category $(\mathbf{CAlg}_k)_{/k}$ spanned by the small objects.

Definition 4.2.2. A formal moduli problem is a functor $X\colon \mathbf{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ satisfying the following conditions:

1. the space $X(k)$ is contractible;
2. if σ is a pullback square:

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & B' \arrow[d, "\phi"] \\ A \arrow[r] & B \end{tikzcd}$$

where ϕ is a small morphism, then $X(\sigma)$ is a pullback diagram in $\mathcal{S}$.

We will denote the ∞-category of formal moduli problem by Moduli.

Example 4.2.3. Let A be a commutative differential graded algebra. We can define a functor

$$\mathrm{Spec}(A)\colon \mathbf{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$$

setting

$$\mathrm{Spec}(A)(B) := \mathrm{Map}_{\mathbf{CAlg}_k^{\mathrm{sm}}}(A, B)$$

Since $\mathrm{Spec}(A)$ preserves limits, it follows that $\mathrm{Spec}(A)$ is a formal moduli problem. moreover, this construction produces a functor $\mathrm{Spec}\colon (\mathbf{CAlg}_k)^{\mathrm{op}} \to \mathrm{Moduli}$.

Lemma 4.2.4. *A morphism $f\colon A \to B$ in $\mathbf{CAlg}_k^{\mathrm{sm}}$ is small if and only if it induces a surjection of commutative rings $\pi_0 A \to \pi_0 B$.*

Proof. Let $K := \mathrm{fib}(f)$. Since A and B are connective, K is (-1)-connective; if moreover $\pi_0 A \to \pi_0 B$ is surjective, then $\pi_{-1}(K) = 0$, so that K is connective as well. Moreover, $\pi_n(K) = 0$ for $k \gg 0$, and $\pi_n(K)$ is always finite dimensional over k. We can therefore argue by induction on $d = \dim_k \pi_*(K)$. If $d = 0$, then $K \simeq 0$ and f is an equivalence, in particular small. Assume now $\pi_*(K) \neq 0$ and let n be the smallest integer such that $\pi_n(K) \neq 0$; if $\mathfrak{m}$ is the maximal ideal of $\pi_0(A)$, then the nilpotent version of Nakayama lemma implies that

$$\mathfrak{m}(\pi_n K) \neq \pi_n(K)$$

In particular, we can choose a nonzero map of $\pi_0(A)$-modules $\phi\colon \pi_n(K) \to k$. Using Theorem 4.1.3 (and shifting everything of -1) one obtains a $(2n+1)$-connective map

$$K \otimes_A B \to \mathbb{L}_{B/A}[-1]$$

This produces an isomorphism

$$\pi_{n+1}\mathbb{L}_{B/A} \simeq \pi_n(K \otimes_A B) \simeq \pi_0(B) \otimes_{\pi_0(A)} \pi_n(K)$$

The map ϕ determines then a map $\pi_{n+1}\mathbb{L}_{B/A} \to k$ which correspond in turn to a map

$$\mathbb{L}_{B/A} \to k[n+1]$$

This map classifies an A-derivation $B \to B \oplus k[n+1]$. Set

$$B' := B \times_{B \oplus k[n+1]} k$$

Then the map f factors as a composition

$$A \xrightarrow{f'} B' \xrightarrow{f''} B$$

The map f'' is elementary, while f' is small by induction, so that we are done. ☐

Proposition 4.2.5. *Let $X\colon \mathbf{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ be a functor. The following conditions are equivalent:*

1. *X satisfies condition 2. in Definition 4.2.2;*

2. *if σ is a pullback square as in Definition 4.2.2, where ϕ is an elementary morphism, then $X(\sigma)$ is a pullback square in $\mathcal{S}$;*

3. *if σ is a pullback square as in Definition 4.2.2 and ϕ is the map $k \to k \oplus k[n]$ for some $n > 0$, then $X(\sigma)$ is a pullback square in $\mathcal{S}$.*

4. *for every pullback diagram σ*

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & B' \arrow[d, "\phi"] \\ A \arrow[r] & B \end{tikzcd}$$

in $\mathbf{CAlg}_k^{\mathrm{sm}}$ such that the maps $\pi_0 B' \to \pi_0 B$ and $\pi_0 A \to \pi_0 B$ are surjective, the induced diagram $X(\sigma)$ is a pullback square.

Proof. The equivalences *1.* $\iff$ *2.* $\iff$ *3.* are consequence of the dual of Proposition 2.1.49. Moreover, *4.* $\Rightarrow$ *3.* is obvious and *1.* $\Rightarrow$ *4.* follows from Lemma 4.2.4. ☐

4.2.2 Tangent complex

Following the analogy with the notion of deformation functor in the classical context, we give the following definition:

Definition 4.2.6. Let $X\colon \mathbf{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ be a formal moduli problem. The tangent space of X is $X(k \oplus k[0]) = X(k[\varepsilon]/(\varepsilon^2))$.

Observe that the tangent space to a formal moduli problem is a space, in the sense of *topological space*. It is possible to recover the algebraic structure on this space using the language of spectra. In fact, we are going to show that each tangent space is an infinite loop space; this implies that it is equipped with an abelian group structure which is well defined up to homotopies.

Recall from Proposition 2.4.5 that we can think to the spectrum $E = \{k \oplus k[n]\}_{n\in\mathbb{N}}$ of Corollary 4.1.11 as a strongly excisive functor

$$E\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathbf{CAlg}_k$$

Under this identification we can prove the following result:

Proposition 4.2.7. *1. for every map $f\colon K \to K'$ in $\mathcal{S}_*^{\mathrm{fin}}$ such that $\pi_0(f)$ is surjective, the induced map $E(K) \to E(K')$ is small;*

2. for every object K of $\mathcal{S}_*^{\mathrm{fin}}$, the object $E(K)$ is small.

Proof. It is clear that *1.* implies *2.* As for this, observe that we can rewrite $f\colon K \to K'$ as a composition

$$K = K_0 \to K_1 \to \cdots \to K_n = K'$$

where K_{i+1} is obtained from K_i by attaching a single cell of dimension n_i; moreover, since $\pi_0(f)$ is surjective, we can assume that $n_i > 0$. We are reduced to show that each $E(K_i) \to E(K_{i+1})$ is a small morphism; we have by assumption the following pushout square:

$$\begin{tikzcd} K_i \arrow[r] \arrow[d] & * \arrow[d] \\ K_{i+1} \arrow[r] & S^n \end{tikzcd}$$

Applying E we obtain a pullback square

$$\begin{tikzcd} E(K_i) \arrow[r] \arrow[d] & k \arrow[d] \\ E(K_{i+1}) \arrow[r] & k \oplus k[n] \end{tikzcd}$$

which shows that $E(K_i) \to E(K_{i+1})$ is an elementary morphism. □

As consequence of this proposition, we can think of E as a functor

$$E\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathbf{CAlg}_k^{\mathrm{sm}}$$

It makes therefore sense to compose a formal moduli problem X with E.

Proposition 4.2.8. *Let $X\colon \mathbf{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ be a formal moduli problem. The composite functor*

$$X \circ E\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$$

is strongly excisive.

Proof. By definition, $X(E(*)) = X(k)$ is weakly contractible. Let

$$\begin{tikzcd} K \arrow[r] \arrow[d] & K' \arrow[d] \\ L \arrow[r] & L' \end{tikzcd}$$

be a pushout square in $\mathcal{S}_*^{\mathrm{fin}}$. Let K'_+ (resp. L'_+) be the union of the connected components of K' (resp. L') having nonempty intersection with the image of $K \to K'$ (resp. of $L \to L'$). We have obvious retractions $K' \to K'_+$ and $L' \to L'_+$; an easy categorical argument shows that the outer square in the following diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d] & K' \arrow[r] \arrow[d] & K'_+ \arrow[d] \\ L \arrow[r] & L' \arrow[r] & L'_+ \end{tikzcd}$$

is a pushout; it follows that also the square on the right is a pushout. Applying $X \circ E$ we obtain

$$\begin{tikzcd} X(E(K)) \arrow[r] \arrow[d] & X(E(K')) \arrow[r] \arrow[d] & X(E(K'_+)) \arrow[d] \\ X(E(L)) \arrow[r] & X(E(L')) \arrow[r] & X(E(L'_+)) \end{tikzcd}$$

and to check that the square on the left is a pullback it will be sufficient to show that both the square on the right and the outer square are pullback. In this way we are reduced to show that

$$\begin{tikzcd} X(E(K)) \arrow[r] \arrow[d] & X(E(K')) \arrow[d] \\ X(E(L)) \arrow[r] & X(E(L'_+)) \end{tikzcd}$$

are pullback whenever the map $\pi_0(L) \to \pi_0(L')$ is surjective, and this is a consequence of the definition of formal moduli problem and Proposition 4.2.7. ☐

Combining Proposition 4.2.8 and Proposition 2.4.5 we conclude that the composition $X \circ E$ is a spectrum (in $\mathcal{S}$).

Definition 4.2.9. Let $X\colon \mathbf{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ be a formal moduli problem. The *tangent complex* of X is defined to be the composite $X \circ E$.

Theorem 4.2.10. *Let $u\colon X \to Y$ be a map of formal moduli problems. If u induces equivalences of tangent complexes X, then X is an equivalence of formal moduli problems.*

Proof. If u is an equivalence, then $u \circ E\colon X \circ E \to Y \circ E$ is an equivalence. For the converse, let $A \in \mathbf{CAlg}_k^{\mathrm{sm}}$ and choose a sequence of elementary morphisms

$$A = A_0 \to A_1 \to \cdots \to A_n \simeq k$$

in $\mathbf{CAlg}_k^{\mathrm{sm}}$. We will show by descending induction on i that the map $u(A_i)\colon X(A_i) \to Y(A_i)$ is an equivalence. If $i = n$, the statement is obvious. Otherwise, we have a diagram

$$\begin{tikzcd} X(A_i) \arrow[r] \arrow[d, "u(A_i)"'] & X(A_{i+1}) \arrow[r] \arrow[d, "u(A_{i+1})"'] & X(k \oplus k[n]) \arrow[d, "u(E_n)"'] \\ Y(A_i) \arrow[r] & Y(A_{i+1}) \arrow[r] & Y(k \oplus k[n]) \end{tikzcd}$$

where the rows are fiber sequences. Now, $u(E_n)$ is an equivalence by hypothesis and $u(A_{i+1})$ is an equivalence by induction. Taking the long exact sequences associated to this diagram, we obtain that $u(A_i)$ is an equivalence as well. ☐

4.2.3 Smooth morphisms

We finally introduce the notion of smoothness for a map of formal moduli problems.

Proposition 4.2.11. *Let $X, Y\colon \mathbf{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ be formal moduli problems and let $u\colon X \to Y$ be a map between them. The following conditions are equivalent:*

1. *for every small map* $\phi\colon A \to B$ *in* $\mathbf{CAlg}_k^{\mathrm{sm}}$*, the map* u *has the RLP with respect to every map* $\mathrm{Spec}(\phi)\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$*;*
2. *for every small map* $\phi\colon A \to B$ *in* $\mathbf{CAlg}_k^{\mathrm{sm}}$*, the natural map*
$$X(A) \to X(B) \times_{Y(B)} Y(A) \tag{4.1}$$
is surjective on connected components;
3. *for every elementary map* $\phi\colon A \to B$ *in* $\mathbf{CAlg}_k^{\mathrm{sm}}$*, the induced map* (4.1) *is surjective on connected components;*
4. *for every* $n > 0$ *the homotopy fibers of* $X(k \oplus k[n]) \to Y(k \oplus k[n])$ *is connected;*
5. *the map of spectra* $X \circ E \to Y \circ E$ *is connective.*

Proof. It follows from the definitions that the elements of $\pi_0(X(B) \times_{Y(B)} Y(A))$ correspond to commutative squares

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

The map $\pi_0(X(A)) \to \pi_0(X(B) \times_{Y(B)} Y(A))$ sends a morphism $\mathrm{Spec}(A) \to X$ to the induced commutative square. Therefore, the map (4.1) is surjective on π_0 if and only if $u\colon X \to Y$ has the RLP with respect to $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. It is therefore clear that *1.* is equivalent to *2.*.

The implications *2.* $\Rightarrow$ *3.* $\Rightarrow$ *4.* are straightforward. Now, let S be the collection of all the small morphisms in $\mathbf{CAlg}_k^{\mathrm{sm}}$ such that the induced map (4.1) is surjective on the connected components. The equivalence of *2.* and *1.* shows that S is closed under composition and under pullbacks; the implications *3.* $\Rightarrow$ *2.* and *4.* $\Rightarrow$ *3.* follow at once.

Finally, the equivalence between *4.* and *5.* follows from the fact that a map of spectra $E \to E'$ is connective if and only if $\Omega^{\infty-n}E \to \Omega^{\infty-n}E'$ has connected homotopy fibers for $n > 0$. ☐

Definition 4.2.12. A map $u\colon X \to Y$ of formal moduli problems is said to be smooth if it satisfies one of the equivalent condition of Proposition 4.2.11.

4.2.4 Smooth hypercovers

We now turn to the existence of smooth hypercovers for a given formal moduli problem.

Definition 4.2.13. Define $\mathrm{Pro}(\mathbf{CAlg}_k^{\mathrm{sm}})$ to be the full subcategory of $\mathrm{Fun}(\mathbf{CAlg}_k^{\mathrm{sm}}, \mathcal{S})^{\mathrm{op}}$ containing all the corepresentable functors and which is closed under filtered colimits. A functor $X\colon \mathbf{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ is said to be prorepresentable if it belongs to $\mathrm{Pro}(\mathbf{CAlg}_k^{\mathrm{sm}})$.

Lemma 4.2.14. *Let* $X\colon \mathbf{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ *be a prorepresentable functor. Then* X *is a formal moduli problem.*

Proof. This follows at once from the fact that filtered colimits in $\mathcal{S}$ are left exact. For a proof of this fact, see [HTT, Proposition 5.3.3.3]. ☐

Lemma 4.2.15. *Let* S *be the collection of all morphisms in* Moduli *of the form* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *where* $A \to B$ *is a small morphism in* $\mathbf{CAlg}_k^{\mathrm{sm}}$*. Let* $f\colon X \to Y$ *be a morphism in* Moduli *and suppose that* f *is a transfinite pushout of morphisms in* $\mathcal{S}$*. If* X *is prorepresentable, then* Y *is prorepresentable.*

Proof. The category $\mathrm{Pro}(\mathbf{CAlg}_k^{\mathrm{sm}})$ is closed under filtered colimits by definition, so that we are immediately reduced to prove the following statement: for every small morphism $\phi\colon A \to B$ in $\mathbf{CAlg}_k^{\mathrm{sm}}$ and every pushout diagram

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r] \arrow[d, "\mathrm{Spec}(\phi)"'] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd}$$

if X is prorepresentable, then Y is prorepresentable as well. However, we have

$$X \simeq \varinjlim_{\beta} \mathrm{Spec}(B_\beta)$$

for some filtered diagram $\{B_\beta\}_{\beta \in I}$ in $(\mathbf{CAlg}_k^{\mathrm{sm}})_{/B}$. Then

$$Y \simeq \varinjlim_{\beta} \mathrm{Spec}(B_\beta) \coprod_{\mathrm{Spec}(B)} \mathrm{Spec}(A)$$

However, one has for every formal moduli problem Z:

$$\begin{aligned} \mathrm{Map}_{\mathrm{Moduli}}\left(\mathrm{Spec}(B_\beta) \coprod_{\mathrm{Spec}(B)} \mathrm{Spec}(A), Z \right) &\simeq Z(B_\beta) \times_{Z(B)} Z(A) \\ &= \simeq Z(B_\beta \times_B A) \end{aligned}$$

It follows by Yoneda lemma that

$$\mathrm{Spec}(B_\beta) \coprod_{\mathrm{Spec}(B)} \mathrm{Spec}(A) \simeq \mathrm{Spec}(B_\beta \times_B A)$$

so that Y is prorepresentable. □

We are now ready to show that every formal moduli problem has a smooth presentation by prorepresentable objects. The result will be a formal consequence of previous lemma and the small object argument.

Proposition 4.2.16. *Let $X \colon \mathrm{CAlg}_k^{\mathrm{sm}} \to \mathcal{S}$ be a formal moduli problem. There exists a simplicial object $X_\bullet$ in $\mathrm{Moduli}_{/X}$ with the following properties:*

1. *each X_n is prorepresentable;*
2. *for each $n \geq 0$ let $M_n(X_\bullet)$ denote the n-th matching object of $X_\bullet$. Then the natural map $X_n \to M_n(X_\bullet)$ is smooth.*

In particular, X is equivalent to the geometric realization $|X_\bullet|$ in $\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm}}, \mathcal{S})$.

Proof. Let S be the class of morphisms $\mathrm{Spec}(\phi) \colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ where A and B are small objects in $\mathbf{CAlg}_k$. Let X be an arbitrary formal moduli problem. Since Moduli is presentable, we can apply Corollary 2.1.73 in order to produce a simplicial object $X_\bullet$ in $\mathrm{Moduli}_{/X}$ in such a way that the maps

$$X_n \to M_n(X_\bullet)$$

have the RLP with respect to the maps in S and the maps

$$L_n(X_\bullet) \to X_n$$

are transfinite pushout of morphisms in S. We claim that this implies that the map $\mathrm{Spec}(k) \to X_n$ is a transfinite pushout of morphisms in S. This can be obtained inductively as follows: let P be the full subcategory of $\mathbf{\Delta}_{\mathbf{n}/}$ spanned by the surjective maps $\mathbf{n} \to \mathbf{m}$; observe that P can be viewed as a poset. For every upward-closed subset P_0, let $Z(P_0)$ be a colimit of the induced diagram

$$N(P_0)^{\mathrm{op}} \longrightarrow N(\mathbf{\Delta})^{\mathrm{op}} \xrightarrow{X_\bullet} \mathrm{Moduli}_{/X} \xrightarrow{\phi} \mathrm{Moduli}$$

where $\phi \colon \mathrm{Moduli}_{/X} \to \mathrm{Moduli}$ is the forgetful functor. Then $Z(\emptyset) \simeq \mathrm{Spec}(k)$ and $Z(P) \simeq \phi(X_n)$. If P_1 is obtained from P_0 by adjoining an element $\alpha \colon \mathbf{n} \to \mathbf{m}$, then the induced map $Z(P_0) \to Z(P_1)$ is a pushout of maps $\phi(L_m(X_\bullet)) \to \phi(X_m)$, so that we proved the claim. At this point, Lemma 4.2.15 implies that each X_n is prorepresentable.

The last statement, follows from the fact that condition *2.* implies that $X_\bullet(A)$ is a hypercovering of $X(A)$ for every A in $\mathbf{CAlg}_k^{\mathrm{sm}}$. □

4.3 Koszul duality functor

We introduced in section 3.3.3 the functor of ∞-categories

$$C^* \colon \mathrm{Lie}_k \to (\mathrm{CAlg}_k^{\mathrm{aug}})^{\mathrm{op}}$$

induced by the cohomological Chevalley - Eilenberg complex. Lemma 3.3.13 shows that Lie_k is presentable. Since C^* preserves small colimits by Proposition 3.3.22, the adjoint functor theorem (Theorem 2.1.71) implies the existence of a right adjoint

$$\mathfrak{D} \colon (\mathrm{CAlg}_k^{\mathrm{aug}})^{\mathrm{op}} \to \mathrm{Lie}_k$$

The main goal of this section is to prove that this functor $\mathfrak{D}$ satisfies a number of technical properties, which will formally imply the main theorem of this mémoire. More specifically, we will show that there exists a full subcategory $\mathcal{C}$ of Lie_k such that the restriction of $\mathfrak{D}$ to $\mathcal{C}$ is not too far from being an equivalence. To make the exposition more readable, we introduce the following definition, to be considered only in this section:

Definition 4.3.1. A differential graded Lie algebra $\mathfrak{g}_*$ is said to be *good* if it is cofibrant and there exists a graded subspace $V_* \subseteq \mathfrak{g}_*$ such that:

1. for every integer $n \in \mathbb{Z}$, the vector space V_n is finite dimensional;
2. $V_n = 0$ if $n \geq 0$;
3. $\mathfrak{g}_*$ is a free Lie algebra over V_*.

Remark 4.3.2. Observe that condition *3.* in the previous definition doesn't imply the cofibrancy of $\mathfrak{g}_*$. In fact, V_* is not required to be a *differential graded* subspace, but simply a graded subspace (otherwise Corollary 3.3.11 would imply the cofibrancy).

Let $\mathcal{C}$ be the full subcategory of Lie_k spanned by those objects that can be represented by a good differential graded Lie algebra. The main result of this section is the following theorem:

Theorem 4.3.3. *The full subcategory $\mathcal{C}$ of Lie_k satisfies the following properties:*

(a) *for every object $\mathfrak{g}_* \in \mathcal{C}$, the unit map $\mathfrak{g}_* \to \mathfrak{D}C^*\mathfrak{g}_*$ is an equivalence;*

(b) *for every $n \geq 1$, there exists an object $\mathfrak{g}_*^{(n)} \in \mathcal{C}$ and an equivalence in $\mathrm{CAlg}_k^{\mathrm{aug}}$*

$$k \oplus k[n] \simeq C^*\mathfrak{g}_*^{(n)}$$

(c) *let $v_n \colon \mathfrak{g}_*^{(n)} \simeq \mathfrak{D}(k \oplus k[n]) \to \mathfrak{D}(k) \simeq 0$. For every pushout diagram*

$$\begin{tikzcd} \mathfrak{g}_*^{(n)} \arrow[r] \arrow[d, "v_n"] & \mathfrak{g}_* \arrow[d] \\ 0 \arrow[r] & \mathfrak{g}'_* \end{tikzcd}$$

if $\mathfrak{g}$ belongs to $\mathcal{C}$, then $\mathfrak{g}'_*$ also belongs to $\mathcal{C}$.

A number of computations is needed to prove this result. We encode the most technical in the following lemma:

Lemma 4.3.4. *let $\mathfrak{g}_*$ be a differential graded Lie algebra over k. If*

1. *for every integer n the vector space $\mathfrak{g}_n$ is finite dimensional, and*
2. *the vector space $\mathfrak{g}_n$ is trivial for $n \geq 0$*

then the unit map $u \colon \mathfrak{g}_* \to \mathfrak{D}C^*(\mathfrak{g}_*)$ is an equivalence in Lie_k.

Proof. See [DAGX, Lemma 2.3.5]. ☐

We are now ready to give the proof of Theorem 4.3.3:

Proof of Theorem 4.3.3. Let $\mathfrak{g}_* \in \mathcal{C}$. In order to verify condition (a), we can assume without loss of generality that $\mathfrak{g}_*$ is good. Let V_* be a graded subspace of $\mathfrak{g}_*$ realizing the conditions of Definition 4.3.1. It follows that, as graded vector space, $\mathfrak{g}_*$ is a direct summand of the tensor algebra

$$\mathcal{T}(V_*) := \bigoplus_{n\geq 0} V_*^{\otimes n}$$

In particular, it follows that each $\mathfrak{g}_n$ is finite dimensional and $\mathfrak{g}_n \simeq 0$ if $n \geq 0$. Condition (a) follows then from Lemma 4.3.4.

Condition (b) is satisfied taking $\mathfrak{g}_*^{(n)} := \mathfrak{f}(k[-n-1])$ (cfr. example 3.3.21). We are left to check condition (c): let $n \leq -2$; we have to compute explicitly the homotopy pushout

$$\begin{tikzcd}
\mathfrak{f}(k[n]) \arrow[r, "\alpha"] \arrow[d, "v"'] & \mathfrak{g}_* \arrow[d] \\
0 \arrow[r] & \mathfrak{g}'_*
\end{tikzcd}$$

A fibrant replacement for v is given by

$$j\colon \mathfrak{f}(k[n]) \to \mathfrak{f}(D^{n+1}(k))$$

It follows from the glueing lemma that the pushout

$$\begin{tikzcd}
\mathfrak{f}(k[n]) \arrow[r, "\alpha"] \arrow[d, "j"'] & \mathfrak{g}_* \arrow[d, "j'"] \\
\mathfrak{f}(D^{n+1}(k)) \arrow[r] & \mathfrak{h}_*
\end{tikzcd}$$

is an explicit model for the homotopy pushout we are interested in. Observe that Corollary 3.3.11 implies that j is a cofibration in $\mathbf{Lie}_k^{\mathrm{dg}}$, so that j' is a cofibration as well; since $\mathfrak{g}_*$ is cofibrant by hypothesis, $\mathfrak{h}_*$ is also cofibrant.

Let $V_* \subseteq \mathfrak{g}_*$ be the subspace realizing the conditions for $\mathfrak{g}_*$ to be good, and let y be the image of a generator of $(D^{n+1}(k))_{n+1}$ in $\mathfrak{h}_{n+1}$. The graded subspace $V'_* \subseteq \mathfrak{h}_*$ generated by V_* and y realizes $\mathfrak{h}_*$ as a good object in $\mathbf{Lie}_k^{\mathrm{dg}}$. ☐

This theorem implies in a formal way several properties of $\mathfrak{D}$:

Proposition 4.3.5. *1.* $\mathfrak{D}(k) \simeq 0$*;*

2. *if* $A_* \in \mathrm{CAlg}_k^{\mathrm{aug}}$ *is of the form* $C^*(\mathfrak{g}_*)$ *with* $\mathfrak{g}_* \in \mathcal{C}$*, then the unit map* $A_* \to C^*\mathfrak{D}(A_*)$ *is an equivalence in* $\mathrm{CAlg}_k^{\mathrm{aug}}$*;*

3. *if* $A_* \in \mathrm{CAlg}_k^{\mathrm{aug}}$ *is small, then* $\mathfrak{D}(A_*) \in \mathcal{C}$ *and* $A_* \to C^*\mathfrak{D}(A_*)$ *is an equivalence in* $\mathrm{CAlg}_k^{\mathrm{aug}}$*;*

4. *if* σ*:*

$$\begin{tikzcd}
A'_* \arrow[r] \arrow[d] & B'_* \arrow[d, "\phi"] \\
A_* \arrow[r] & B_*
\end{tikzcd}$$

is a homotopy pullback in $\mathrm{CAlg}_k^{\mathrm{aug}}$ *where* A*,* B *and* ϕ *are small, then* $\mathfrak{D}(\sigma)$ *is a pushout.*

Proof. 1. By adjoint nonsense we have $C^*(0) \simeq k$; since $0 \in \mathcal{C}$ it follows from condition (a) of Theorem 4.3.3 that the unit map $0 \to \mathfrak{D}C^*(0) \simeq \mathfrak{D}(k)$ is an equivalence.

2. Let $A_* = C^*(\mathfrak{g}_*)$ with $\mathfrak{g}_* \in \mathcal{C}$. Then the unit map (in $(\mathrm{CAlg}_k^{\mathrm{aug}})^{\mathrm{op}}$)
$$\eta \colon C^*\mathfrak{D}C^*(\mathfrak{g}_*) \to A_* = C^*(\mathfrak{g}_*)$$
has a left homotopy inverse induced by the edge $\varepsilon \colon \mathfrak{g}_* \to \mathfrak{D}C^*(\mathfrak{g}_*)$. Since $\mathfrak{g}_*$ belongs to $\mathcal{C}$, condition (a) of Theorem 4.3.3 implies that $\mathfrak{g}_* \to \mathfrak{D}C^*(\mathfrak{g}_*)$ is an equivalence. It follows that $C^*(\varepsilon)$ is an equivalence as well, so that η is forced to be an equivalence.

3. Let A_* be a small commutative differential graded algebra. Choose a sequence of elementary morphisms
$$A_* = A_*^{(0)} \to A_*^{(1)} \to \cdots \to A_*^{(n)} \simeq k$$
in $\mathbf{CAlg}_k$. We will show that $A_*^{(i)} \simeq C^*(\mathfrak{h}_*^{(i)})$ for some $\mathfrak{h}_*^{(i)} \in \mathcal{C}$ by descending induction on i. Point (b) will imply immediately that the unit map $A_*^{(i)} \to C^*\mathfrak{D}(A_*^{(i)})$ is an equivalence. Moreover, we have the following equivalence in Lie_k:
$$\mathfrak{D}(A_*^{(i)}) \simeq \mathfrak{D}C^*(\mathfrak{h}_*^{(i)}) \simeq \mathfrak{h}_*^{(i)}$$
thanks to condition (a) of Theorem 4.3.3, completing the proof of point *3.*.

   If $i = n$, the assertion follows from point *1.* Assume now $i < n$; since $A_*^{(i)} \to A_*^{(i+1)}$ is elementary, there exists an integer $n > 0$ and a pullback diagram σ:
$$\begin{tikzcd} A_*^{(i)} \arrow[r] \arrow[d] & k \arrow[d, "\phi"] \\ A_*^{(i+1)} \arrow[r, "\psi"'] & k \oplus k[n] \end{tikzcd}$$
Form the following homotopy pushout τ in $\mathbf{Lie}_k^{\mathrm{dg}}$:
$$\begin{tikzcd} \mathfrak{D}(k \oplus k[n]) \arrow[r] \arrow[d] & \mathfrak{D}(A_*^{(i+1)}) \arrow[d] \\ \mathfrak{D}(k) \arrow[r] & X \end{tikzcd}$$
The unit transformation for the adjunction induces a morphism of diagrams
$$\xi \colon \sigma \to C^*(\tau)$$
in $\mathbf{CAlg}_k^{\mathrm{aug}}$. Observe that $A_*^{(n+1)}$ belongs to the essential image of $C^*|_{\mathcal{C}}$ by the inductive hypothesis, while k and $k \oplus k[n]$ satisfy the same hypothesis because of condition (b) of Theorem 4.3.3. Since both σ and $C^*(\tau)$ are pullback, it follows that ξ is an equivalence, so that $A_*^{(i+1)} \simeq C^*(X)$. Condition (d) of Theorem 4.3.3 implies that $A_*^{(i+1)} \in \mathcal{C}$.

4. The dual version of Proposition 2.1.49 implies that the class of morphisms ϕ making the statement true is closed under composition; we are therefore reduced to deal with the case where ϕ is elementary. Again, Proposition 2.1.49 and its dual version, show that we can limit ourselves to the situation where ϕ is the map $k \to k \oplus k[n]$. The same argument used in point *3.* shows that the diagram σ
$$\begin{tikzcd} A'_* \arrow[r] \arrow[d] & k \arrow[d] \\ A_* \arrow[r] & k \oplus k[n] \end{tikzcd}$$
is equivalent to a diagram of the form $C^*(\tau)$ where τ is a diagram in $\mathcal{C}$ which is a pushout in Lie_k. It follows that
$$\mathfrak{D}(\sigma) \simeq \mathfrak{D}C^*(\tau) \simeq \tau$$
so that we obtain the thesis.

□

Corollary 4.3.6. *Let $j\colon \mathrm{Lie}_k \to \mathrm{Fun}(\mathrm{Lie}_k^{\mathrm{op}}, \mathcal{S})$ be the Yoneda embedding. For every $\mathfrak{g}_* \in \mathrm{Lie}_k$ the composition*

$$\mathrm{CAlg}_k \xrightarrow{\mathfrak{D}} \mathrm{Lie}_k^{\mathrm{op}} \xrightarrow{j(\mathfrak{g}_*)} \mathcal{S}$$

is a formal moduli problem. In particular we obtain a functor

$$\Psi\colon \mathrm{Lie}_k \to \mathrm{Moduli}$$

In particular, for every differential graded Lie algebra $\mathfrak{g}_*$ we obtain a spectrum

$$\Psi(\mathfrak{g}_*) \circ E\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$$

This construction determines a functor

$$e\colon \mathrm{Lie}_k \to \mathrm{Sp}(\mathcal{S})$$

Proposition 4.3.7. *The functor e preserves small sifted colimits and it reflects equivalences.*

Proof. Let $\theta\colon \mathrm{Lie}_k \to \mathrm{Mod}_k$ be the forgetful functor. Consider the spectrum $S = \{k[-n]\}_{n \geq 0}$ in $(\mathbf{Vect}_k^{\mathrm{dg}})^{\mathrm{op}}$ and introduce the other "forgetful" functor

$$F\colon \mathrm{Mod}_k \to \mathrm{Sp}(\mathcal{S})$$

defined by

$$F(M_*) = \mathrm{Map}_{\mathbf{Vect}_k^{\mathrm{dg}}}(S(-), M_*)\colon \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$$

This is strongly excisive, so it defines a spectrum in $\mathcal{S}$. Moreover, since F is a functor between stable $(\infty, 1)$-categories and since it trivially commutes with finite limits, it follows from [HA, Proposition 1.1.4.1] that it commutes with finite colimits as well; since every object $k[-n]$ is compact, it follows also that it commutes with filtered colimits. In conclusion, F commutes with every colimit (because arbitrary coproducts can be obtained via binary coproducts and filtered colimits).

Observe also that the functor F is conservative. In fact, we have

$$\mathrm{Map}_{\mathbf{Vect}_k^{\mathrm{dg}}}(k[-n], M_*) = \mathrm{Map}_{\mathbf{Vect}_k^{\mathrm{dg}}}(k, \tau_{\leq 0}(M_*[n]))$$

and an explicit model for this mapping space is obtained by applying the Dold-Kan functor to $\tau_{\leq 0}(M_*[n])$ and then forgetting the module structure. Since the Dold-Kan functor is part of a Quillen equivalence, we see that F reflects equivalences (because if $F(u)$ is an equivalence, then $\tau_{\leq n}(u)$ is an equivalence for every $n \geq 0$, so that u is an equivalence as well).

Fix a differential graded Lie algebra $\mathfrak{g}_*$ representing an object in Lie_k; we see that a model for $e(\mathfrak{g}_*)$ is given by

$$\mathrm{Map}_{\mathbf{Lie}_k^{\mathrm{dg}}}(\mathfrak{D}(E), \mathfrak{g}_*)$$

However, Theorem 4.3.3.(b) implies that $\mathfrak{D}(E)$ is the spectrum

$$\{\mathfrak{f}(k[-n-1])\}_{n \geq 0}$$

in $\mathrm{Lie}_k^{\mathrm{op}}$. Therefore we have

$$\mathrm{Map}_{\mathbf{Lie}_k^{\mathrm{dg}}}(\mathfrak{D}(E), \mathfrak{g}_*) = \mathrm{Map}_{\mathbf{Vect}_k^{\mathrm{dg}}}(S(-)[-1], \Theta(\mathfrak{g}_*))$$

It follows that e is given by $(F \circ \theta)[-1]$. Since θ is trivially conservative, the thesis follows from Lemma 3.3.13. □

Corollary 4.3.8. *For every small commutative differential graded algebra $A_* \in \mathbf{CAlg}_k^{\mathrm{sm}}$, $\mathfrak{D}(A_*)$ is a compact object in the ∞-category Lie_k.*

Proof. Choose a sequence of elementary morphisms

$$A = A_*^{(0)} \to A_*^{(1)} \to \cdots \to A_*^{(n)} \simeq k$$

in $\mathbf{CAlg}_k$. We will use descending induction on i in order to show that $\mathfrak{D}(A_*^{(i)}$ is a compact object in Lie_k. When $i = n$, this follows from point *1.* of Proposition 4.3.5. Assume now that $i < n$ and that $\mathfrak{D}(A_*^{(i+1)})$ is compact. Since $A_*^{(i)} \to A_*^{(i+1)}$ is elementary, there exists an integer $n > 0$ and a pullback daigram σ:

$$\begin{tikzcd} A_*^{(i)} \arrow[r] \arrow[d] & k \arrow[d] \\ A_*^{(i+1)} \arrow[r] & k \oplus k[n] \end{tikzcd}$$

in $\mathbf{CAlg}_k$. Point *4.* of Proposition 4.3.5 implies that $\mathfrak{D}(\sigma)$ is a pushout square in Lie_k. It will be therefore sufficient to show that $\mathfrak{D}(A_*^{(i+1)})$, $\mathfrak{D}(k)$ and $\mathfrak{D}(k \oplus k[n])$ are compact object in Lie_k; using the inductive hypothesis, we are left to check that $\mathfrak{D}(k \oplus k[n])$ is a compact object. However, the functor corepresented by $\mathfrak{D}(k \oplus k[n])$ is obtained as the composition

$$\mathrm{Lie}_k \xrightarrow{e} \mathrm{Spectra}(\mathcal{S}) \xrightarrow{\Omega^{\infty-n}} \mathcal{S}$$

Proposition 4.3.7 implies that e commutes with filtered colimits, and the same is true for the functor $\Omega^{\infty-n}$. It follows that $\mathfrak{D}(k \oplus k[n])$ is a compact object in Lie_k. □

4.4 The main theorem

We are ready to prove the main theorem:

Theorem 4.4.1. *The functor* $\Psi \colon \mathrm{Lie}_k \to \mathrm{Moduli}$ *is an equivalence of* ∞*-categories.*

Proof. Recall that Ψ is given by the formula

$$\Psi(\mathfrak{g}_*)(A_*) := \mathrm{Map}_{\mathrm{Lie}_k}(\mathfrak{D}(A_*), \mathfrak{g}_*)$$

This makes clear that Ψ preserves small limits in Lie_k. Corollary 4.3.8 implies that Ψ preserves filtered colimits as well, so that it is accessible. The adjoint functor theorem (Theorem 2.1.71) implies therefore that Ψ has a left adjoint Φ. Using Corollary 2.1.65 we are reduced to show that

(i) Ψ reflects equivalences;

(ii) the unit transformation $u \colon \mathrm{id}_{\mathrm{Moduli}} \to \Psi \circ \Phi$ is an equivalence.

We begin with (i). Assume that $f \colon \mathfrak{g}_* \to \mathfrak{h}_*$ is a morphism in Lie_k and $\Psi(f)$ is an equivalence. In particular, we obtain for every $n \geq 0$ a homotopy equivalence

$$\begin{aligned} \mathrm{Map}_{\mathrm{Lie}_k}(\mathfrak{D}(k \oplus k[n]), \mathfrak{g}_*) &\simeq \Psi(\mathfrak{g}_*)(k \oplus k[n]) \\ &\simeq \Psi(\mathfrak{h}_*)(k \oplus k[n]) \\ &\simeq \mathrm{Map}_{\mathrm{Lie}_k}(\mathfrak{D}(k \oplus k[n]), \mathfrak{h}_*) \end{aligned}$$

We deduce that $e(f) \colon e(\mathfrak{g}_*) \to e(\mathfrak{h}_*)$ is an equivalence; Proposition 4.3.7 implies now that f is an equivalence as well, showing point (i).

Let us turn to the proof of (ii). Let X be a formal moduli problem; if we can show that the map

$$\theta \colon X \circ E \to (\Psi \circ \Phi)(X) \circ E \simeq e(\Phi(X))$$

is an equivalence of spectra, Theorem 4.2.10 will imply that the unit map $u \colon X \to (\Psi \circ \Phi)(X)$ is an equivalence as well. Choose a simplicial object $X_\bullet$ of $\mathrm{Moduli}_{/X}$ satisfying the conditions of Proposition 4.2.16. For every object $A \in \mathbf{CAlg}_k^{\mathrm{sm}}$, the simplicial space $X_\bullet(A)$ is a hypercovering of $X(A)$, so that the induced map $|X_\bullet(A)| \to X(A)$ is a homotopy equivalence. It follows that X is a colimit for the simplicial diagram $X_\bullet$ in $\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm}}, \mathcal{S})$ and hence in Moduli. Similarly, $X \circ E$ is equivalent to the geometric realization $|X_\bullet \circ E|$ in the ∞-category of spectra. Since Φ commutes with small colimits and e preserves sifted colimits, we obtain

$$e(\Phi(X)) \simeq e(\Phi(|X_\bullet|)) \simeq |e(\Phi(X_\bullet))|$$

It follows that θ is the geometric realization of a simplicial morphism

$$\theta_\bullet \colon X_\bullet \circ E \to e(\Phi(X_\bullet))$$

We are therefore reduced to show that each θ_n is an equivalence. Repeating the argument above, we see that this is equivalent to show that u induces an equivalence $X_n \to (\Psi \circ \Phi)(X_n)$. In this way, we are left to check condition (ii) in the case where X is prorepresentable. Since Φ and Ψ commute with filtered colimits, we can further reduce the analysis to the case $X = \mathrm{Spec}(A)$ for some $A \in \mathbf{CAlg}_k^{\mathrm{sm}}$. Since

$$\Phi(\mathrm{Spec}(A)) = \mathfrak{D}(A)$$

we only have to show that for each $B \in \mathbf{CAlg}_k^{\mathrm{sm}}$ the map

$$\mathrm{Map}_{\mathbf{CAlg}_k}(A, B) \to \mathrm{Map}_{\mathrm{Lie}_k}(\mathfrak{D}(B), \mathfrak{D}(A)) \simeq \mathrm{Map}_{\mathbf{CAlg}_k}(A, C^*\mathfrak{D}(B))$$

is a homotopy equivalence. However, this follows immediately from point *3.* of Proposition 4.3.5. $\square$

Appendix A

Simplicial Sets

As general principle, the reader is assumed to be familiar with the theory of simplicial sets. In this chapter, we collect some of the results which aren't easy to track in the classical literature (as [GZ67], [May69] or [GoJa]). In practice, we collect here a number of technical tools which have been proved very useful in writing down the proofs contained in this mémoire; more specifically, we will show that the geometric realization reflects colimits (Corollary A.1.3), we will give sufficient conditions for a set of monomorphisms to be saturated (Propositions A.2.8 and A.2.6), and we will develop a relative version of the join operation.

A.1 Geometric realization

This section is devoted to prove that the geometric realization reflects colimits. This is an useful property that allows to prove in an almost straightforward way that certain diagrams of simplicial sets are pushouts, while a direct argument would require much more work and combinatorial observations.

We recall, without proving it, the following crucial theorem:

Theorem A.1.1. *The geometric realization functor* $|\cdot|\colon \mathbf{sSet} \to \mathbf{CGHaus}$ *commutes with finite limits and with colimits. Moreover it reflects isomorphisms.*

Proof. [GZ67, Ch. III.3]. □

We can obtain our goal as a direct consequence of the following observation:

Lemma A.1.2. *Let* $\mathcal{C}$ *be a cocomplete category and let* $F\colon \mathcal{C} \to \mathcal{D}$ *be a functor preserving colimits. If in addition* F *reflects isomorphisms, then* F *reflects colimits.*

Proof. Let $\mathbf{I}$ be a small category and let $G\colon \mathbf{I} \to \mathcal{C}$ be an $\mathbf{I}$-diagram in $\mathcal{C}$. Let $\eta\colon G \to \Delta_x$ be a natural transformation, where $\Delta_x\colon \mathbf{I} \to \mathcal{C}$ is the constant diagram at an element $x \in \mathrm{Ob}(\mathcal{C})$; assume that $F(\eta)\colon F \circ G \to F \circ \Delta_x = \Delta_{F(x)}$ is a colimit diagram in $\mathcal{D}$. Since $\mathcal{C}$ is cocomplete, let $\varphi\colon G \to \Delta_y$ be a colimit diagram for F in $\mathcal{C}$; let $f\colon y \to x$ be the unique map induced by the universal property of colimits. It follows that $F(f)$ is an isomorphism, so that f was an isomorphism to begin with. □

Corollary A.1.3. *The geometric realization functor* $|\cdot|\colon \mathbf{sSet} \to \mathbf{CGHaus}$ *reflects colimits.*

Proof. It's consequence of Theorem A.1.1 and Lemma A.1.2, since $\mathbf{sSet}$ is cocomplete. □

A.2 The formalism of saturated sets

In this section we develop a flexible theory of anodyne extensions. The theory presented here is a slight generalization of the classical one (due to P. Gabriel and M. Zisman) and it is needed in Chapter 2 in order to deal with different classes of fibrations. However, we limit ourselves in the framework of simplicial sets, avoiding needless generalizations to wider classes of categories.

Definition A.2.1. A class of *monomorphisms* $\mathcal{M}$ in **sSet** is said to be saturated if the following holds:

1. $\mathcal{M}$ contains all the isomorphisms;
2. $\mathcal{M}$ is closed under retracts;
3. $\mathcal{M}$ is closed under pushouts;
4. $\mathcal{M}$ is closed under transfinite compositions.

Lemma A.2.2. *The intersection of an arbitrary family of saturated set of monomorphisms is still saturated.*

Proof. This is obvious. ☐

Lemma A.2.3. *Let $f\colon K \to L$ be a morphism of simplicial sets and let $\mathcal{M}$ be the class of monomorphisms in* **sSet** *having the LLP with respect to f. Then $\mathcal{M}$ is saturated.*

Proof. This is a straightforward check. Some detail can be found in [GoJa, Lemma I.4.1]. ☐

Corollary A.2.4. *Let $\mathcal{S}$ be a set of arrows in* **sSet** *and let $\mathcal{M}$ be the class of monomorphisms in* **sSet** *having the LLP with respect to every arrow in S. Then $\mathcal{M}$ is saturated.*

Proof. This is an immediate consequence of previous two lemmas. ☐

Notation A.2.5. Let $F\colon \mathbf{sSet} \times \mathbf{sSet} \to \mathbf{sSet}$ be a bifunctor. If $f\colon A \to B$ and $g\colon X \to Y$ are morphisms of simplicial sets, denote by $\mathfrak{A}_F(f,g)$ the object

$$F(A,Y) \coprod_{F(A,X)} F(B,X)$$

and by $\widetilde{F}(f,g)$ the induced morphism

$$\widetilde{F}(f,g)\colon \mathfrak{A}_F(f,g) \to F(B,Y)$$

Proposition A.2.6. *Let $\mathcal{S}_1$, $\mathcal{S}_2$ be classes of monomorphisms in* **sSet** *and assume that $\mathcal{S}_2$ is saturated. Let $F\colon$* **sSet** $\times$ **sSet** $\to$ **sSet** *be a bifunctor such that for every simplicial set K the functor $F(-,K)$ commutes with colimits. Let $\mathcal{M}$ be the class of monomorphism f in* **sSet** *such that $\widetilde{F}(f,g) \in \mathcal{S}_2$ for every $g \in \mathcal{S}_1$. Assume that $\mathcal{M}$ satisfies the following property:*

(SR) *given $f\colon A \to B$ and $g\colon X \to Y$, the map $F(f,Y)\colon F(A,Y) \to F(B,Y)$ is in $\mathcal{S}_2$ whenever f is in $\mathcal{M}$ and g is in $\mathcal{S}_1$.*

Then $\mathcal{M}$ is saturated.

Proof. 1. *Isomorphisms.* Let $f\colon A \to B$ be an isomorphism and let $g\colon X \to Y$ be a morphism in $\mathcal{S}_1$. Then we have the following pushout diagram:

$$\begin{tikzcd}
F(A,X) \arrow[r, "F(A,g)"] \arrow[d, "F(f,X)"'] & F(A,Y) \arrow[d, equal] \arrow[dr, bend left, "F(f,Y)"] & \\
F(B,X) \arrow[r, "F(f^{-1},Y)"] & F(A,Y) \arrow[r, "F(f,Y)"] & F(B,Y)
\end{tikzcd}$$

so that $\mathfrak{A}_F(f,g) = F(A,Y)$ and $\widetilde{F}(f,g) = F(f,Y)$. Since this is an isomorphism and $\mathcal{S}_2$ is saturated, it follows that $f \in \mathcal{M}$.

2. *Retractions.* Start with a retraction diagram

$$\begin{tikzcd}
C \arrow[r, "v"] \arrow[d, "f"'] & A \arrow[r, "u"] \arrow[d, "\varphi"'] & C \arrow[d, "f"'] \\
D \arrow[r, "r"'] & B \arrow[r, "s"'] & D
\end{tikzcd}$$

with $uv = \mathrm{id}_C$, $sr = \mathrm{id}_D$ and $\varphi \in \mathcal{M}$. For every $g\colon X \to Y$ in $\mathcal{S}_1$ it is straightforward to check that the following is a retraction diagram:

$$\begin{tikzcd}
\mathfrak{A}_F(f,g) \arrow[r] \arrow[d] & \mathfrak{A}_F(\varphi,g) \arrow[r] \arrow[d] & \mathfrak{A}_F(f,g) \arrow[d] \\
F(D,Y) \arrow[r] & F(B,Y) \arrow[r] & F(D,Y)
\end{tikzcd}$$

Since $\widetilde{F}(\varphi,g) \in \mathcal{S}_2$ by hypothesis and since $\mathcal{S}_2$ is saturated, we deduce that $\widetilde{F}(f,g) \in \mathcal{S}_2$, so that $f \in \mathcal{M}$.

3. *Pushouts.* Consider a pushout square

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\varphi"'] & C \arrow[d, "f"] \\
B \arrow[r] & D
\end{tikzcd}$$

where $\varphi \in \mathcal{M}$. For every $g\colon X \to Y$ in $\mathcal{S}_1$ we can form the following commutative diagram (the maps are the ones induced functorially by F):

$$\begin{tikzcd}
F(A,X) \arrow[rr] \arrow[dr] \arrow[dd] & & F(B,X) \arrow[dr] \arrow[dd] & & & \\
& F(C,X) \arrow[rr] \arrow[dd] & & F(D,X) \arrow[dd] & & \\
F(A,Y) \arrow[rr] \arrow[dr] & & \mathfrak{A}_F(\varphi,g) \arrow[rr] \arrow[dr] & & F(B,Y) \arrow[dr] & \\
& F(C,Y) \arrow[rr] & & \mathfrak{A}_F(f,g) \arrow[rr] & & F(D,Y)
\end{tikzcd}$$

By hypothesis the two squares

$$\begin{tikzcd}
F(A,X) \arrow[r] \arrow[d] & F(B,X) \arrow[d] \\
F(C,X) \arrow[r] & F(D,X)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A,Y) \arrow[r] \arrow[d] & F(B,Y) \arrow[d] \\
F(C,Y) \arrow[r] & F(D,Y)
\end{tikzcd}$$

are pushouts. It follows that the outer square in

$$\begin{tikzcd}
F(A,X) \arrow[r] \arrow[d] & F(A,Y) \arrow[r] \arrow[d] & F(C,Y) \arrow[d] \\
F(B,X) \arrow[r] & \mathfrak{A}_F(\varphi,g) \arrow[r] & \mathfrak{A}_F(f,g)
\end{tikzcd}$$

is a pushout. Abstract nonsense shows immediately that also the right square is a pushout; for the same reason we deduce that

$$\begin{tikzcd}
\mathfrak{A}_F(\varphi,g) \arrow[r] \arrow[d, "\widetilde{F}(\varphi,g)"'] & \mathfrak{A}_F(f,g) \arrow[d, "\widetilde{F}(f,g)"] \\
F(B,Y) \arrow[r] & F(D,Y)
\end{tikzcd}$$

is a pushout square. Since $\widetilde{F}(\varphi,g) \in \mathcal{S}_2$ and since $\mathcal{S}_2$ is saturated, it follows that $\widetilde{F}(f,g) \in \mathcal{S}_2$, that is $f \in \mathcal{M}$.

4. *Transfinite composition.* Let $\{f_\beta \colon A_\beta \to A_{\beta+1}\}_{\beta<\lambda}$ be a λ-sequence of morphisms in $\mathcal{M}$ and let
$$\mathrm{in}_\beta \colon A_\beta \to A_\lambda$$
be the canonical morphisms toward the colimit. For any $g \colon X \to Y$ in $\mathcal{S}_1$ we see that
$$F(f_\lambda, Y) \colon F(A_0, Y) \to F(A_\lambda, Y)$$
is the composition of the sequence
$$\{F(f_\beta, Y) \colon F(A_\beta, Y) \to F(A_{\beta+1}, Y)\}_{\beta<\lambda}$$
By hypothesis, we have $F(f_\beta, Y) \in \mathcal{S}_2$. Stability of $\mathcal{S}_2$ under pushouts implies immediately that the natural maps
$$\mathfrak{A}_F(\mathrm{in}_\beta, g) \to \mathfrak{A}_F(\mathrm{in}_{\beta+1}, g)$$
are in $\mathcal{S}_2$. Since the map
$$\widetilde{F}(\mathrm{in}_0, g) \colon \mathfrak{A}_F(\mathrm{in}_0, g) \to F(A_\infty, Y)$$
can be written as countable composition of those maps, it follows that $\widetilde{F}(\mathrm{in}_0, g)$ lies in $\mathcal{S}_2$, proving that $\mathrm{in}_0 \in \mathcal{M}$.

$\square$

Lemma A.2.7. *In the notations and hypothesis of Proposition A.2.6, if $\mathcal{S}_1$ contains the morphisms $\varphi_Y \colon \emptyset \to Y$ for every simplicial set Y and the functor $F(A, \emptyset) = \emptyset$ for every simplicial set A, then $\mathcal{M}$ satisfies the hypothesis* (SR).

Proof. Simply observe that under these hypothesis we have $\widetilde{F}(f, \varphi_Y) = F(f, Y)$. $\square$

The same result holds interchanging the role of the variables of the functor F:

Proposition A.2.8. *Let $\mathcal{S}_1$, $\mathcal{S}_2$ be classes of monomorphisms in* **sSet** *and assume that $\mathcal{S}_2$ is saturated. Let $F \colon$* **sSet** $\times$ **sSet** $\to$ **sSet** *be a bifunctor such that for every simplicial set K the functor $F(K, -)$ commutes with colimits. Let $\mathcal{M}$ be the class of monomorphism f in* **sSet** *such that $\widetilde{F}(g, f) \in \mathcal{S}_2$ for every $g \in \mathcal{S}_1$. Assume that $\mathcal{M}$ satisfies the following property:*

(SL) *given $f \colon A \to B$ and $g \colon X \to Y$, the map $F(Y, f) \colon F(Y, A) \to F(Y, B)$ is in $\mathcal{S}_2$ whenever f is in $\mathcal{M}$ and g is in $\mathcal{S}_1$.*

Then $\mathcal{M}$ is saturated.

Lemma A.2.9. *In the notations and hypothesis of Proposition A.2.8, if $\mathcal{S}_1$ contains the morphisms $\varphi_Y \colon \emptyset \to Y$ for every simplicial set Y and the functor $F(\emptyset, A) = \emptyset$ for every simplicial set A, then $\mathcal{M}$ satisfies the hypothesis* (SL).

We omit the proofs since they are identical to the previous ones.

A.3 Relative join of simplicial sets

In order to deal efficiently with the combinatoric underlying the whole theory of quasicategories, several technical tools are needed. For example, in Chapter 2 we made an extensive use of fibrations and anodyne extensions; another technical construction that we used through all the chapter was the join of simplicial sets. This section is devoted to this operation and to its properties. We actually propose a relative version of the join, generalizing a little the one given in [HTT]; this is needed in order to deal with ∞-adjunctions because it produces a neat way to construct ∞-correspondences.

A.3.1 Ideas and motivations

Before plunging into the technical exposition, we want to explain the notion of correspondence in classical category theory, as motivation. In fact, one should think to a relative join of simplicial sets as an ∞-categorical generalization of this basic construction. The arising technicalities are due, as usual, to the combinatoric of simplicial sets.

Definition A.3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A correspondence from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$.

Given a correspondence F from $\mathcal{C}$ to $\mathcal{D}$ one can define a new category $\mathcal{C} \star^F \mathcal{D}$ in the following way: the set of objects of this category will be the disjoint union of $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Ob}(\mathcal{D})$. Moreover, we set

$$\mathrm{Hom}_{\mathcal{C}\star^F\mathcal{D}}(X,Y) := \begin{cases} \mathrm{Hom}_{\mathcal{C}}(X,Y) & \text{if } X,Y \in \mathrm{Ob}(\mathcal{C}) \\ \mathrm{Hom}_{\mathcal{D}}(X,Y) & \text{if } X,Y \in \mathrm{Ob}(\mathcal{D}) \\ F(X,Y) & \text{if } X \in \mathrm{Ob}(\mathcal{C}) \text{ and } Y \in \mathrm{Ob}(\mathcal{D}) \\ \emptyset & \text{otherwise} \end{cases}$$

The composition is defined in the obvious way, using the compositions of $\mathcal{C}$, of $\mathcal{D}$ and the functoriality of F. The new category $\mathcal{C} \star^F \mathcal{D}$ is equipped with a natural functor

$$p\colon \mathcal{C} \star^F \mathcal{D} \to \mathbf{1}$$

sending $\mathcal{C}$ (as subcategory of $\mathcal{C}\star^F\mathcal{D}$) into 0, $\mathcal{D}$ into 1 and every arrow starting from an object of $\mathcal{C}$ and landing in an object of $\mathcal{D}$ to $0 < 1$. Conversely, given a category $\mathscr{M}$ equipped with a functor

$$p\colon \mathscr{M} \to \mathbf{1}$$

in such a way that $\mathscr{M}_{\{0\}} \simeq \mathcal{C}$ and $\mathscr{M}_{\{1\}} \simeq \mathcal{D}$, then we can define a correspondence from $\mathcal{C}$ to $\mathcal{D}$.

It is the last description that we can easily generalize to the ∞-categorical context. However, in practice it might be difficult to explicitly define a ∞-correspondence. The relative join of simplicial set is to be thought as a simplicial equivalent of the first construction we gave. We will show in fact, that associated to every relative join there is a simplicial correspondence (which is an ∞-correspondence under reasonable assumptions).

A.3.2 The construction

Notation A.3.2. If J is a linearly ordered finite set and $I, I' \subset J$ are subsets, we write

$$I < I'$$

to mean that $i < i'$ for each $i \in I$ and each $i' \in I'$.

Notation A.3.3. Let K be a simplicial set and let J be a nonempty linearly ordered finite set. Thinking J as a category, it is isomorphic to the category $\mathbf{n} \in \mathbf{\Delta}$, where $n = \#J$ is the cardinality of J. We let $K(J) := K(\mathbf{n})$. If, instead, J is empty we define $K(J) := \{*\}$.

Notation A.3.4. Let $k, h \in \mathbb{N}$ and set $n = k + h + 1$. Given a simplicial set S and simplexes $\sigma \in S_k$, $\tau \in S_h$, we denote by $S_n^{\sigma,\tau} = \mathrm{Hom}_{\sigma,\tau}(\Delta^n, S)$ the set of n-simplexes ω of S such that

$$\omega|_{\{0,\dots,k\}} = \sigma, \qquad \omega|_{\{k+1,\dots,n\}} = \tau$$

If J is a nonempty linearly ordered finite set we will denote by $S^{\sigma,tau}(J)$ the set $S_n^{\sigma,\tau}$, where $n = \#J$.

Let us fix three simplicial sets K, L, S and maps

$$K \xrightarrow{f} S \xleftarrow{g} L$$

We define a new simplicial set $K\,{}_f{\star}_g\,L$ in the following way: given a linearly ordered finite set J, set:

$$(K\,{}_f{\star}_g\,L)(J) := \coprod_{\substack{J = J_1 \sqcup J_2 \\ J_1 < J_2}} \coprod_{\substack{(\sigma,\tau)\in \\ K(J_1)\times K(J_2)}} S^{f(\sigma),g(\tau)}(J)$$

where, if $J_1 = \emptyset$ or $J_2 = \emptyset$ we set $S^{f(\sigma),g(\tau)}(J) := \{*\}$.

To define the action on maps, fix a morphism $\varphi\colon I \to J$. If $J = J_1 \sqcup J_2$ is a partition of J with $J_1 < J_2$, we obtain (functorially) a partition of $I = I_1 \sqcup I_2$, where

$$I_1 = \varphi^{-1}(J_1), \qquad I_2 = \varphi^{-1}(J_2)$$

Let $\varphi_k\colon I_k \to J_k$ the induced map for $k = 1, 2$. For every $(\sigma, \tau) \in K(J_1) \times L(J_2)$ write

$$\varphi_1^*\sigma := (\varphi_1)_K^*(\sigma) \in K(I_1), \qquad \varphi_2^*\tau := (\varphi_2)_L^*(\tau) \in L(I_2)$$

Observe that:

$$f(\varphi_1^*\sigma) = (\varphi_1)_S^*(f(\sigma)), \qquad g(\varphi_2^*\tau) = (\varphi_2)_S^*(g(\tau))$$

Lemma A.3.5. *The map $\varphi_S^*\colon S(J) \to S(I)$ restricts to a well defined map*

$$(S^{f(\sigma),g(\tau)}(J) \to S^{f(\varphi_1^*\sigma),g(\varphi_2^*\tau)}(I)$$

Proof. Denote by $i_k\colon I_k \to I$ and by $j_k\colon J_k \to J$ the natural inclusions, so that one has

$$\varphi \circ i_k = j_k \circ \varphi_k$$

If $\omega \in S^{f(\sigma),g(\tau)}$ then by hypothesis

$$\omega \circ j_1 = f(\sigma), \qquad \omega \circ j_2 = g(\tau)$$

It follows that

$$\begin{aligned}\varphi_S^*(\omega) \circ i_1 &= \omega \circ \varphi \circ i_1 = \omega \circ j_1 \circ \varphi_1 \\ &= (\varphi_1)_S^*(\omega \circ j_1) = (\varphi_1)_S^*(f(\sigma))\end{aligned}$$

and similarly one obtains

$$\varphi_S^*(\omega) \circ i_2 = (\varphi_2)_S^*(g(\tau))$$

completing the proof of the lemma. □

The previous lemma and the universal property of the coproducts imply the existence of a canonical map

$$K_f\star_g L)(J) \to (K_f\star_g K)(I)$$

It is a standard exercise in classical category theory to show that this assignment is functorial. As consequence we obtain a well defined simplicial set $K_f\star_g L$.

Definition A.3.6. Given maps of simplicial sets $f\colon K \to S$, $g\colon L \to S$, the simplicial set $K_f\star_g L$ is said to be the *relative join* of K and L with respect to S.

If $S = \Delta^0$, then we will denote $K_f\star_g L$ by $K \star L$ and we will refer to it as the (absolute) join of K and L.

Notation A.3.7. If the maps f and g are clear from the context, we will write $K \star_S L$ to denote $K_f\star_g L$.

Lemma A.3.8. *Given maps of simplicial sets $f\colon K \to S$, $g\colon L \to S$ there are naturally induced maps $\varphi = \varphi_{f,g}\colon K \to K \star_S L$ and $\psi = \psi_{f,g}\colon L \to K \star_S L$ and $\pi = \pi_{f,g}\colon K \star_S L \to S$ making the diagram*

$$\begin{tikzcd} K \arrow[r, "\varphi"] \arrow[dr, "f"'] & K \star_S L \arrow[d, "\pi"] & L \arrow[l, "\psi"'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

commutative.

Proof. For every $\sigma \in K_n$ define

$$\varphi(\sigma) := f(\sigma) \in S^{f(\sigma),*} \subseteq \coprod_{(\sigma,*)\in K_n \times L(\emptyset)} S^{f(\sigma),*}$$

It is straightforward to check that this is a morphism of simplicial sets. Naturality in K is also obvious. In a similar way one can define $\psi\colon L \to K \star_S L$. The definition of π is even more straightforward: one has inclusions

$$S_n^{f(\sigma),g(\tau)} \subset S_n$$

inducing morphisms $(K \star_S L)_n \to S_n$, which are obviously compatible with face and degeneracy maps. ☐

A.3.3 The ∞-categorical properties

To use the relative join in dealing with ∞-categories, it is important to have some stability condition, guaranteeing that $K \star_S L$ is a quasicategory. We propose the following result, generalizing [HTT, Proposition 1.2.8.3].

Proposition A.3.9. *If $f\colon K \to S$ and $g\colon L \to S$ are inner fibrations of simplicial sets and S is a quasicategory, then $K \star_S L$ is a quasicategory.*

Proof. Start with a horn inclusion $\alpha\colon \Lambda_i^n \to K \star_S L$, where $0 < i < n$. Since S is a quasicategory, we can choose a map $\beta\colon \Delta^n \to S$ making the diagram

$$\begin{tikzcd} \Lambda_i^n \arrow[r, "\alpha"] \arrow[d, tail, "i"'] & K \star_S L \arrow[d] \\ \Delta^n \arrow[r, "\beta"'] & S \end{tikzcd}$$

commutative. If α factors through $\varphi\colon K \to K \star_S L$, using the inner fibrancy of $f = \pi \circ \varphi$ we obtain a lifting $h\colon \Delta^n \to K$ such that

$$h \circ i = \alpha, \qquad f \circ h = \beta$$

so in particular the lifting exists. We can proceed in a symmetric way if α factors through $\psi\colon L \to K \star_S L$.

Assume now that α doesn't factor through φ nor ψ. Then let k be the minimal integer such that $\alpha(k)$ belongs to the image of L in $K \star_S L$. It follows that α induces morphisms $\Delta^k \to K$ and $\Delta^{n-k-1} \to L$. These morphisms correspond in turn to an n-simplex of $\Delta^n \to K \star_S L$, extending the horn inclusion α. ☐

We introduced the relative join as a tool to produce ∞-correspondences. We show now that the relative join $K \star_S L$ comes equipped with a natural morphism $K \star_S L \to \Delta^1$ defining a simplicial correspondence.

Proposition A.3.10. *Let $f\colon K \to S$ and $g\colon L \to S$ be morphisms of simplicial sets. There exists a natural morphism $p\colon K \star_S L \to \Delta^1$ such that $p^{-1}(0) \simeq f(K)$ and $p^{-1}(1) \simeq g(L)$.*

Proof. For an n-simplex ω in $K \star_S L$ we can canonically associate a pair $(\sigma, \tau) \in K(I) \times L(J)$ for suitable ordered sets I and J. If $I, J \neq \emptyset$, denote by $\varphi_{I,J}\colon \mathbf{n} \to \mathbf{1}$ the unique morphism such that

$$\varphi_{I,J}^{-1}(0) = \{0, \dots, \#I\}$$

Let ι_1 be the only non-degenerate 1-simplex of Δ^1. Define:

$$p(\omega) := \begin{cases} 0 & \text{if } J = \emptyset \\ 1 & \text{if } I = \emptyset \\ \varphi_{I,J}^* \iota_1 & \text{otherwise} \end{cases}$$

(where we identify 0, 1 and ι_1 with their degeneracies if necessary). It is straightforward to check that p is a morphism of simplicial sets, as well as the conditions on the fibers. ☐

A.3.4 The absolute join

Despite being only a particular case of the relative join, the absolute join is perhaps the more frequent operation used in chapter 2. For this reason, it is convenient to list here some of its properties.

Before beginning, we specialize the construction of the relative join to the absolute context. If K and L are simplicial sets, $K \star L$ is defined by its value on a finite linearly ordered set J. The involved formula simplifies as follows:

$$(K \star L)(J) := \coprod_{\substack{J = J_1 \sqcup J_2 \\ J_1 < J_2}} K(J_1) \times L(J_2)$$

Proposition A.3.11. *If S and S' are quasicategories, then $S \star S'$ is a quasicategory as well.*

Proof. This is an immediate corollary of Proposition A.3.9. □

Example A.3.12. Let $K, L \in \mathbf{sSet}$ and let $\alpha\colon \Delta^1 \to K \star L$. This means that α belongs to one among K_1, L_1 or $K_0 \times L_0$. In the first two cases:

$$d_i^{K \star L}\alpha = d_i^K \alpha, \qquad d_i^{K \star L}\alpha = d_i^L \alpha$$

while in the last we have $\alpha = (x, y)$, where $x \in K_0$ and $y \in L_0$, and

$$d_0^{K \star L}\alpha = x, \qquad d_1^{K \star L}\alpha = y$$

In particular, we see that if $\omega\colon \Delta^n \to K \star L$ is such that $\omega(m) \in K_0$ then for every $k \leq m$ we must have $\omega(k) \in K_0$: otherwise, we would have a 1-simplex $\alpha\colon \Delta^1 \to K \star L$ with $d_0\alpha \in L$ and $d_1\alpha \in K$, which is impossible.

Lemma A.3.13.

1. *$\star$ defines a bifunctor $\mathbf{sSet} \times \mathbf{sSet} \to \mathbf{sSet}$;*
2. *given $K \in \mathbf{sSet}$, the functors $K \star -\colon \mathbf{sSet} \to \mathbf{sSet}$ and $- \star K\colon \mathbf{sSet} \to \mathbf{sSet}$ commute with colimits;*
3. *$\Delta^i \star \Delta^j \simeq \Delta^{i+j+1}$;*
4. *$\partial\Delta^i \star \Delta^0 \simeq \Lambda^{i+1}_{i+1}$.*

Proof. The first point is straightforward. The second point is an easy consequence of the cartesian closedness of $\mathbf{sSet}$. □

Appendix B

Model categories

Model categories play a key role throughout this whole mémoire. Nevertheless, a full exposition of the subject is far beyond the purpose of this work. The reader is referred to [DS95] for a very readable introduction to the subject, and to the books [Hov99] and [Hir03] for a more complete exposition. This appendix is supposed to collect some of the main results needed throughout the mémoire; the focus is mainly on existence theorems for model structures.

B.1 Combinatorial model structures

Model categories form an invaluable tool in dealing with higher homotopies, but it is often technical to check that a given families of morphisms define a model structure; moreover, it is often hard to get a good understanding of the whole set of fibrations (or cofibrations). There are, however, classes of model categories which are very well understood. For example, one usually works with model categories which are *cofibrantly generated*. Here we recall the definition:

Definition B.1.1. A *cofibrantly generated model category* is a model category $\mathcal{M}$ such that:

1. there exists a set I of maps that permits the small objects argument and such that a map is a trivial fibration if and only if it has the RLP with respect to all the maps in I;

2. there exists a set J of maps that permits the small objects argument and such that a map is a fibration if and only if it has the RLP with respect to all the maps in J.

In a cofibrantly generated model category the functoriality of the factorization follows immediately from the small object argument. Combinatorial model categories represent a small variation on this idea: one adds hypotheses on the behaviour of the underlying category $\mathcal{M}$. Precisely, one gives the following definition:

Definition B.1.2. A *combinatorial model category* is a cofibrantly generated model category $\mathcal{M}$ which is moreover presentable.

Combinatorial model categories have a number of good properties and are stable under a lot of categorical operations. The key result in proving such stability results is the following recognition theorem:

Theorem B.1.3. *Let $\mathcal{M}$ be a presentable category and let $W \subset \mathbf{Arr}(\mathcal{M})$ be an accessible subcategory of $\mathbf{Arr}(\mathcal{M})$. Let I be a small set of morphisms in $\mathcal{M}$ and assume that*

1. *W satisfies the two-out-of-three-axiom;*

2. *the set $\mathrm{inj}(W)$ is contained in W;*

3. *the intersection $W \cap \mathrm{cof}(I)$ is closed under pushouts and transfinite composition.*

Then C is a combinatorial model category with weak equivalences W, cofibrations $\mathrm{cof}(I)$ and fibrations $\mathrm{inj}(W \cap \mathrm{cof}(I))$.

Proof. See [Bar07, Proposition 1.7]. □

Theorem B.1.4. *Let $\mathcal{M}$ be a combinatorial model category and let $\mathcal{C}$ be a small category. Then there exist two combinatorial model structure over $\mathcal{M}^{\mathcal{C}}$:*

- *the projective model structure, where weak equivalences and fibrations are defined objectwise;*
- *the injective model structure, where weak equivalences and cofibrations are defined objectwise.*

Proof. See [HTT, Prop. A.2.8.2]. □

B.2 Existence criteria

B.2.1 Transfer principle

I first learned of the theorem I am going to prove in the article [GS06]. Many different versions have appeared in the literature, but the argument is essentially due to Quillen. I am giving here a general enough statement to make it work in many cases of interest for this mémoire; if the reader is interested in other formulations, he can look at [SS00, Lemma 2.3], [GoJa, Theorem II.6.8], [Hin97, Theorem 2.2.1].

Theorem B.2.1. *Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ be an adjoint pair and suppose that $\mathcal{C}$ is a cofibrantly generated model category. Let I and J be chosen sets of generating cofibrations and acyclic cofibrations, respectively. Define a morphism $f\colon X \to Y$ to be a weak equivalence or a fibration if $G(f)$ is so. Suppose further that*

1. *the functor G commutes with sequential colimits;*
2. *a map in $\mathcal{D}$ with the LLP with respect to every fibration is a weak equivalence.*

Then $\mathcal{D}$ becomes a cofibrantly generated model category. Furthermore the sets $\{F(i) \mid i \in I\}$ and $\{F(j) \mid j \in J\}$ generate the cofibrations and the acyclic cofibrations of $\mathcal{D}$ respectively.

Proof. It is straightforward to check axioms **MC1.** to **MC3.** The proof of **MC4.** relies on **MC5.**, hence we begin by this last one. We will prove that every morphism $f\colon X \to Y$ in $\mathcal{D}$ ca be factored as

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

where j is a cofibration and q is a trivial fibration. The proof is a variation of the small object argument: we construct inductively a family of objects Z_n together with cofibrations $j_n\colon Z_n \to Z_{n+1}$ and maps $q_n\colon Z_n \to Y$ such that

$$q_n = q_{n+1} \circ j_{n+1}$$

Set $Z_0 = X$, $j_0 = \mathrm{id}_X$ and $q_0 = f$. Define Z_{n+1} in the following way: let $\mathcal{S}_{n+1}$ be the set of triples (φ, j, ψ) where $j\colon A \to B$ is a morphism in J and φ and ψ are morphisms in $\mathcal{C}$ making the diagram

$$\begin{tikzcd} F(A) \arrow[r, "\varphi"] \arrow[d, "F(j)"'] & Z_n \arrow[d, "q_n"] \\ F(B) \arrow[r, "\psi"'] & Y \end{tikzcd}$$

commutative. Define now $j_{n+1}\colon Z_n \to Z_{n+1}$ to be the pushout of the following diagram

$$\begin{tikzcd} \coprod_{\mathcal{S}_{n+1}} F(A) \arrow[r] \arrow[d, "\mathcal{S}^{n+1}F(j)"'] & Z_n \arrow[d, "j_{n+1}"] \\ \coprod_{\mathcal{S}_{n+1}} F(B) \arrow[r] & Z_{n+1} \end{tikzcd}$$

A formal adjunction argument shows that each $F(j)$ has to be a cofibration in $\mathcal{D}$; moreover, the class of cofibration is saturated (because defined as intersection of saturated classes), so that the morphism $j_{n+1}\colon Z_n \to Z_{n+1}$ is a cofibration by construction. Finally, the universal property of pushout produces a map $q_{n+1}\colon Z_{n+1} \to Y$. Set

$$Z := \varinjlim_n Z_n$$

We obtain in this way a factorization of $f\colon X \to Y$ as $q \circ j$, where $j\colon X \to Z$ is the inclusion of $Z_0 = X$ into the colimit. This map is a cofibration because cofibrations are saturated. We are left to show that q is a trivial fibration, i.e. that $G(q)$ is a trivial fibration in $\mathcal{C}$. Consider the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & G(Z) \arrow[d, "G(q)"] \\ B \arrow[r] & G(Y) \end{tikzcd}$$

Since $\mathcal{C}$ is cofibrantly generated, both A and B are small. Since G commutes with sequential colimits, this lifting problem becomes equivalent to

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & G(Z_n) \arrow[d, "G(q_n)"] \\ B \arrow[r] & G(Y) \end{tikzcd}$$

and a standard adjunction argument shows that this problem is equivalent to

$$\begin{tikzcd} F(A) \arrow[r] \arrow[d, "F(j)"'] & Z_n \arrow[d, "q_n"] \\ F(B) \arrow[r] & Y \end{tikzcd}$$

which has solution by construction. In a similar way, but using I instead of J we can prove that every morphism $f\colon X \to Y$ in $\mathcal{D}$ can be factored as $q \circ i$ where i is a cofibration with the LLP with respect to every fibration and q is a fibration.

We finally prove **MC4**. The LLP of cofibrations with respect to trivial fibrations holds exactly by definition of cofibration. Let $f\colon A \to B$ be a trivial cofibration and factor it as

$$A \xrightarrow{i} C \xrightarrow{q} B$$

where i is a cofibration with the LLP with respect to every fibration and q is a fibration. By hypothesis, i is a trivial cofibration; it follows by **MC2**. that q is a weak equivalence. Therefore we can solve the following lifting problem:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & C \arrow[d, "q"] \\ B \arrow[r, equal] \arrow[ur, dotted, "s"] & B \end{tikzcd}$$

and so we can express f as a retract of i. It follows that f is has the LLP with respect to all fibrations. □

Remark B.2.2. Keeping track of the cardinals involved in the proof, it is possible to state a more precise version of Theorem B.2.1. Namely, if we require the category $\mathcal{C}$ to be cofibrantly generated with respect to a cardinal κ, then we will only need to require that the right adjoint commutes with κ-filtered colimits.

Condition *2.* is often hard to verify in practice. We give a standard argument, due to Quillen, to check in practice when this condition holds:

Proposition B.2.3. *Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ be an adjunction and assume that $\mathcal{C}$ has a model structure. Define a morphism $f\colon X \to Y$ in $\mathcal{D}$ to be a weak equivalence or a fibration if $G(f)$ is so. If*

1. *every object in $\mathcal{D}$ is fibrant, and*
2. *every object in $\mathcal{D}$ has a path object,*

then every cofibration with the LLP with respect to every fibration is a weak equivalence.

Proof. Let $f\colon A \to B$ be a cofibration with the LLp with respect to every fibration; since A is fibrant by hypothesis, we can choose a retraction $r\colon B \to A$ of f because in the following diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[d] \\ B \arrow[r] \arrow[ur, dotted, "r"] & * \end{tikzcd}$$

the lifting exists by assumption. This expresses f as retract of $f \circ r$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[r, "r"] \arrow[d, "f \circ r"'] & A \arrow[d, "f"'] \\ B \arrow[r, equal] & B \arrow[r, equal] & B \end{tikzcd}$$

so that we are reduced to show that $f \circ r$ is a weak equivalence. Let

$$B \xrightarrow{i} B^I \xrightarrow{p} B \times B$$

be a path object for B. In the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[r, "i"] & B^I \arrow[d, "p"] \\ B \arrow[rr, "(\mathrm{id}_B, f \circ r)"'] \arrow[urr, dotted, "h"] & & B \times B \end{tikzcd}$$

the lifting exists by assumption. Let $p_k\colon B \times B \to B$ for $k = 1, 2$ be the two natural projection; then $p_k \circ p$ is a weak equivalence for the 2 out of 3 axiom, so that

$$p_1 \circ p \circ h = \mathrm{id}_B$$

implies that h is a weak equivalence. Therefore

$$p_2 \circ p \circ h = f \circ r$$

implies that $f \circ r$ is a weak equivalence as well, completing the proof. □

Finally, we state a slightly more general version of the previous theorem.

Theorem B.2.4. *Let $\mathcal{C}$ be a cofibrantly generated model category. Assume that $\{F_i\colon \mathcal{D} \rightleftarrows \mathcal{C}\colon G_i\}_{i \in I}$ be a family of adjoint functors. Define a morphism $f\colon X \to Y$ in $\mathcal{D}$ to be a weak equivalence or a fibration if $G_i(f)$ is so for every $i \in I$. Suppose further that:*

1. *every functor G_i commutes with filtered colimits;*
2. *the functors G_i takes the saturation of the collection of all maps $F_j A \to F_j B$ arising from maps $A \to B$ in the generating family for the cofibrations of $\mathcal{C}$ and elements $j \in I$ to cofibrations of $\mathcal{C}$;*
3. *a map in $\mathcal{D}$ with the LLP with respect to every fibration is a weak equivalence.*

Then the weak equivalences and the fibrations we introduced in $\mathcal{D}$ define a cofibrantly generated model structure on $\mathcal{D}$.

Proof. The proof is essentially unchanged. The only part that should be rewritten is the proof of the factorization axiom, but the needed changes are straightforward. See [GoJa, Theorem II.6.8] for an outline. □

B.2.2 Categories of monoids

Theorem B.2.5. *Let $\mathcal{C}$ be a combinatorial monoidal model category. Assume that either every object of $\mathcal{C}$ is cofibrant, or that $\mathcal{C}$ is a symmetric monoidal model category which satisfies the monoid axiom. Defining a morphism $f\colon A \to B$ in $\mathrm{Alg}(\mathcal{C})$ to be a weak equivalence or a fibration if it is so in $\mathcal{C}$, the category $\mathrm{Alg}(\mathcal{C})$ inherits a combinatorial model structure.*

Proof. See [SS00, Theorem 4.1] □

Proposition B.2.6. *Let $\mathcal{C}$ be a combinatorial monoidal model category and let $\mathbf{I}$ be a small category such that $N(\mathbf{I})$ is sifted. Assume either that every object of $\mathcal{C}$ is cofibrant, or that $\mathcal{C}$ satisfies the following pair of conditions:*

1. *the monoidal structure on $\mathcal{C}$ is symmetric and $\mathcal{C}$ satisfies the monoid axiom;*
2. *the model category $\mathcal{C}$ is left proper and the class of cofibrations in $\mathcal{C}$ is generated by cofibrations between cofibrant objects.*

Let W be the collection of weak equivalences in $\mathcal{C}^c$ and W' the collection of weak equivalences in $\mathrm{Alg}(\mathcal{C})^c$. Then the forgetful functor

$$N(\mathrm{Alg}(\mathcal{C})^c)[W'^{-1}] \to N(\mathcal{C}^c)[W^{-1}]$$

preserves $N(\mathbf{I})$-indexed colimits.

Proof. See [HA, Lemma 4.1.4.13]. □

B.2.3 Categories of simplicial objects

We now turn to another interesting application of Theorem B.2.1. We fix a category $\mathcal{C}$ and we look for the category of simplicial objects in $\mathcal{C}$. For sake of completeness, we recall the definition:

Definition B.2.7. Let $\mathcal{C}$ be any category. A simplicial object in $\mathcal{C}$ is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$. The (functorial) category of simplicial objects in $\mathcal{C}$ is denoted $\mathrm{s}\mathcal{C}$.

The first remarkable observation is that $\mathrm{s}\mathcal{C}$ is always enriched over **sSet**. We follow closely the construction given by Quillen [Qui67, p. II.1.7], but the idea goes back at least to D. Kan. To build this enrichment we will employ the machinery of ends. We recall here a basic result:

Lemma B.2.8. *Let $\mathcal{B}$ be a category with products. Let $\mathcal{A}$ be a small category and let $F, G\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{B}$ be functors. The following statements hold:*

1. *if $\mathcal{A}$ is a discrete category then*
$$\int_{A \in \mathcal{A}} F(A,A) \simeq \prod_{A \in \mathcal{A}} F(A)$$
2. *denote by $F \times G$ the functor*
$$F \times G\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \xrightarrow{(F,G)} \mathcal{B} \times \mathcal{B} \xrightarrow{- \times -} \mathcal{B}$$
Then
$$\int_{A \in \mathcal{A}} (F \times G)(A,A) \simeq \int_{A \in \mathcal{A}} F(A,A) \times \int_{A \in \mathcal{A}} G(A,A)$$

Proof. 1. Denote by C the product

$$\prod_{A \in \mathcal{A}} F(A,A)$$

and let $\pi_A\colon C \to F(A,A)$ be the canonical projection. Since $\mathcal{A}$ is discrete, we see that $\{\pi_A\colon C \to F(A,A)\}_{A \in \mathcal{A}}$ is a wedge over F. The universal property of the product shows that this wedge is also a universal wedge.

2. Recall the construction of the decomposition category of MacLane: if $\mathcal{C}$ is a category, the objects of $\mathcal{C}^{\S}$ are $\mathrm{Ob}(\mathcal{C}) \sqcup \mathrm{Arr}(\mathcal{C})$; we denote by $A^{\S}$ and $f^{\S}$ objects and arrows of $\mathcal{C}$ when we think of them as objects of $\mathcal{C}^{\S}$. Moreover, for each arrow $f\colon A \to B$ in $\mathcal{C}$ we have two arrows $A^{\S} \to f^{\S}$ and $B^{\S} \to f^{\S}$. Adding the identities of objects makes $\mathcal{C}^{\breve{g}}$ into a category. If $H\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is a functor, then we obtain an induced functor $H^{\S}\colon \mathcal{C}^{\S} \to \mathcal{D}$, and H has an end if and only if $H^{\S}$ has limit, in which case the two of them coincide (cfr. [Mac71, Proposition IX.5.1]).

   In our case, we simply observe that $(F \times G)^{\S} = F^{\S} \times G^{\S}$, so that the lemma is a trivial consequence of the fact that limits commutes with limits.

□

Theorem B.2.9. *For every category $\mathcal{C}$, the category $\mathrm{s}\mathcal{C}$ is enriched over* **sSet**.

Proof. Fix two simplicial objects $X, Y \in \mathrm{Ob}(\mathrm{s}\mathcal{C})$ and a simplicial set $K\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$. We want to define a bifunctor

$$M = M_{X,K,Y}\colon \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta} \to \mathbf{Set}$$

We define it on objects as:

$$M(\mathbf{m}, \mathbf{n}) := K_n \cdot \mathrm{Hom}_{\mathcal{C}}(X_n, Y_m) = \coprod_{\sigma \in K_n} \mathrm{Hom}_{\mathcal{C}}(X_n, Y_m)$$

If $f\colon \mathbf{n}_1 \to \mathbf{n}_2$ and $g\colon \mathbf{m}_1 \to \mathbf{m}_2$ are arrows in $\mathbf{\Delta}$ we define

$$M(f, g)\colon K_{n_2} \cdot \mathrm{Hom}_{\mathcal{C}}(X_{n_2}, Y_{m_2}) \to K_{n_1} \cdot \mathrm{Hom}_{\mathcal{C}}(X_{n_1}, Y_{m_1})$$

to be the natural morphism induced by

$$f_K^*\colon K_{n_2} \to K_{n_1}, \qquad f_X^*\colon X_{n_2} \to X_{n_1}, \qquad g_Y^*\colon Y_{m_2} \to Y_{m_1}$$

Consider the end of the functor M:

$$\mathrm{Map}(X \times K, Y) := \int_{\mathbf{n} \in \mathbf{\Delta}} M(\mathbf{n}, \mathbf{n})$$

If $\alpha\colon X' \to X$, $\beta\colon Y \to Y'$ and $\gamma\colon K' \to K$ are natural transformation we obtain another natural transformation

$$M_{\alpha,\gamma,\beta}\colon M_{X,K,Y} \to M_{X',K',Y'}$$

The universal property of ends produces now a unique map (cfr. [Mac71, Proposition IX.7.1])

$$\mathrm{Map}(\alpha \times \gamma, \beta)\colon \int_{\mathbf{n} \in \mathbf{\Delta}} M_{X,K,Y}(\mathbf{n}, \mathbf{n}) \to \int_{\mathbf{n} \in \mathbf{\Delta}} M_{X',K',Y'}(\mathbf{n}, \mathbf{n})$$

and the uniqueness shows that this assignment is functorial. We therefore get a well–defined functor

$$\mathrm{Map}\colon (\mathrm{s}\mathcal{C})^{\mathrm{op}} \times \mathbf{sSet}^{\mathrm{op}} \times \mathrm{s}\mathcal{C} \to \mathbf{Set}$$

We build the simplicial enrichment using this functor, via the formula:

$$\mathrm{Hom}_{\mathrm{s}\mathcal{C}}(X, Y; \mathbf{sSet}) := \mathrm{Map}(X \times \Delta^{\bullet}, Y)$$

First of all let us check the formula:

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^0, \mathrm{Hom}_{\mathrm{s}\mathcal{C}}(X, Y; \mathbf{sSet})) = \mathrm{Hom}_{\mathrm{s}\mathcal{C}}(X, Y) \tag{B.1}$$

For every $\mathbf{n} \in \mathrm{Ob}(\mathbf{\Delta})$ we can define a morphism

$$\omega_n\colon \mathrm{Hom}_{\mathrm{s}\mathcal{C}}(X, Y) \to M(\mathbf{n}, \mathbf{n}) = M_{X,\Delta^0,Y}(\mathbf{n}, \mathbf{n})$$

by

$$\omega_n(\varphi) := \varphi_n$$

Let $f\colon \mathbf{n} \to \mathbf{m}$ be a morphism in $\mathbf{\Delta}$. Then the diagram

$$\begin{tikzcd}
& M(\mathbf{n},\mathbf{n}) \arrow[dr, "M(1,f)"] & \\
\mathrm{Hom}_{s\mathcal{C}}(X,Y) \arrow[ur, "\omega_n"] \arrow[dr, "\omega_m"'] & & M(\mathbf{n},\mathbf{m}) \\
& M(\mathbf{m},\mathbf{m}) \arrow[ur, "M(f,1)"'] &
\end{tikzcd}$$

commutes. In fact,

$$M(1,f)(\omega_n(\varphi)) = \varphi_n \circ f_Y^* = f_X^* \circ \varphi_n = M(f,1)(\omega_n(\varphi))$$

Moreover, if

$$\{\eta_n \colon \{*\} \to M(\mathbf{n},\mathbf{n})\}_{\mathbf{n}\in\mathbf{\Delta}}$$

is a wedge over M, the commutativity of the same square of above forces

$$\eta_n \circ f_Y^* = f_X^* \circ \eta_n$$

which means that $\{\eta_n\}_{\mathbf{n}\in\mathbf{\Delta}}$ defines a natural transformation $\eta\colon X \to Y$. This is obviously enough to assert that $\{\omega_n\}_{\mathbf{n}\in\mathbf{\Delta}}$ defines a universal wedge from $\mathrm{Hom}_{s\mathcal{C}}(X,Y)$ to M, that is

$$\mathrm{Hom}_{s\mathcal{C}}(X,Y) \simeq \int_{\mathbf{n}\in\mathbf{\Delta}} M(\mathbf{n},\mathbf{n})$$

showing that (B.1) holds.

Now let us define the composition for the simplicially enriched hom-sets. Given simplicial objects X, Y, Z in $s\mathcal{C}$ and any simplicial set K we consider the functor

$$M_{X,K,Y} \times M_{Y,K,Z} \colon \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta} \to \mathbf{Set}$$

Since

$$M_{X,K,Y}(\mathbf{n},\mathbf{m}) \times M_{Y,K,Z}(\mathbf{n},\mathbf{m}) \simeq K_n \cdot (\mathrm{Hom}_{\mathcal{C}}(X_n, Y_m) \times \mathrm{Hom}_{\mathcal{C}}(Y_n, Z_m))$$

we obtain (using composition in $\mathcal{C}$) a dinatural transformation

$$\alpha\colon M_{X,K,Y} \times M_{Y,K,Z} \to M_{X,K,Z}$$

Lemma B.2.8 shows that this dinatural transformation induces a morphism

$$\mathrm{Map}(X \times K, Y) \times \mathrm{Map}(Y \times K, Z) \to \mathrm{Map}(X \times K, Z)$$

and the functoriality of this construction yields the desired composition map

$$\mathrm{Hom}_{s\mathcal{C}}(X, Y; \mathbf{sSet}) \times \mathrm{Hom}_{s\mathcal{C}}(Y, Z; \mathbf{sSet}) \to \mathrm{Hom}_{s\mathcal{C}}(X, Z; \mathbf{sSet})$$

The universal properties employed to construct this composition map implies immediately the associativity; the identity is obviously defined using the identification (B.1), and again the universal properties appearing in the construction imply that the unit diagram commutes. $\square$

Previous theorem is particularly interesting because it doesn't make use of any assumption on the category $\mathcal{C}$. Adding some hypothesis we obtain an enrichment with better properties, as tensor and cotensor. The result is the following:

Theorem B.2.10. *If $\mathcal{C}$ is a category with coproducts, then $s\mathcal{C}$ is enriched with tensor over* **sSet**. *If $\mathcal{C}$ has limits, then $s\mathcal{C}$ is enriched with cotensor over* **sSet**.

Sketch of the proof. Given a simplicial set K and a simplicial object $X \in \mathrm{s}\mathcal{C}$, define

$$(X \otimes K)_n := \coprod_{\sigma \in K_n} X_n$$

If $\varphi \colon \mathbf{n} \to \mathbf{m}$ is an arrow in $\mathbf{\Delta}$, define

$$\varphi^*_{X\otimes K} \colon (X \otimes K)_m \to (X \otimes K)_n$$

as the map

$$\coprod_{\sigma \in K_m} \mathrm{in}_{\varphi^*_K(\sigma)} \varphi^*_X$$

This defines a new simplicial object in $\mathrm{s}\mathcal{C}$, which realizes the tensor. More details can be found in [Qui67, Proposition II.1.2]. □

The second main result concerns, instead, the model structure of $\mathrm{s}\mathcal{C}$. First of all recall the following definitions:

Definition B.2.11. Let $\mathcal{C}$ be a category. A morphism $f \colon X \to Y$ is said to be an effective epimorphism if it has kernel pair and it is the quotient of its kernel pair, that is if the diagram

$$\begin{tikzcd} X \times_Y X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

is a coequalizer.

Definition B.2.12. Let $\mathcal{C}$ be a category. An object $P \in \mathrm{Ob}(\mathcal{C})$ is said to be projective if $f_* \colon \mathrm{Hom}_{\mathcal{C}}(P, X) \to \mathrm{Hom}_{\mathcal{C}}(P, Y)$ is surjective for every effective epimorphism $f \colon X \to Y$.

Definition B.2.13. A category $\mathcal{C}$ has enough projectives if for each object X there is a projective object P and an effective epimorphism $P \to X$.

Lemma B.2.14. *Let $\mathcal{C}$ be any category with pullbacks and let (T, μ, η) be a monad over $\mathcal{C}$. For any T-algebra (A, h) the map $h \colon T(A) \to A$ is an effective epimorphism.*

Proof. Let (A, h) be an algebra for the monad T. We claim that $T(A)$ is projective in $\mathcal{C}^T$ and that $h \colon T(A) \to A$ is an effective epimorphism. First of all, observe that $\mathcal{C}^T$ has pullbacks. An arrow is an effective epimorphism if and only if it is a regular epimorphism, and as trivial consequence we see that every split epimorphism is effective. However, the unit axiom says that

$$h \circ T\eta_A = \mathrm{id}_A$$

i.e. h is a split epimorphism. □

Theorem B.2.15. *Let $\mathcal{C}$ be a category with finite limits and enough projectives. Define a map f in $\mathrm{s}\mathcal{C}$ to be a fibration (resp. a weak equivalence) if $\mathrm{Hom}_{\mathrm{s}\mathcal{C}}(P, f; \mathbf{sSet})$ is a fibration (resp. a weak equivalence) for each projective object P. If moreover $\mathcal{C}$ satisfies one of the following extra conditions:*

1. *every object of $\mathrm{s}\mathcal{C}$ is fibrant;*

2. *$\mathcal{C}$ is closed under inductive limits and has a set of small projective generators;*

then $\mathrm{s}\mathcal{C}$ has simplicial model structure.

Proof. The original proof can be found in [Qui67, Thm. II.4.4]. □

In general it is not straightforward to characterize projective objects in a category. As consequence, the description of fibrations and weak equivalences given in the previous theorem might not be as explicit as one would hope. However, whenever there is a set of projective generators, we can significantly reduce the number of needed checks.

Proposition B.2.16. *Let $\mathcal{C}$ be a category with finite limits and a set of projective generators $\{P_i\}_{i\in I}$ such that for every object X in $\mathcal{C}$ there is an effective epimorphism*

$$\coprod_{j\in J} P_{i_j} \to X$$

Then a map $f\colon X \to Y$ in $\mathrm{s}\mathcal{C}$ is a fibration (resp. a weak equivalence) in the sense of Theorem B.2.15 if and only if $\mathrm{Hom}_{\mathrm{s}\mathcal{C}}(P_i, f; \mathbf{sSet})$ is a fibration (resp. a weak equivalence) for each index $i \in I$.

Proof. If a map is a fibration (resp. a weak equivalence) then the map $\mathrm{Hom}_{\mathrm{s}\mathcal{C}}(P_i, f; \mathbf{sSet})$ is a fibration (resp. a weak equivalence) by definition.

Conversely, assume that $\mathrm{Hom}_{\mathrm{s}\mathcal{C}}(P_i, f; \mathbf{sSet})$ is a fibration (resp. a weak equivalence) for all indexes $i \in I$; let P be any projective object in $\mathcal{C}$. Choose an effective epimorphism

$$r\colon \coprod_{j\in J} P_{i_j} \to P$$

Since P is projective, this map has a section s. We obtain in this way a retraction diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{s}\mathcal{C}}(P, X; \mathbf{sSet}) \arrow[r] \arrow[d] & \prod \mathrm{Hom}_{\mathrm{s}\mathcal{C}}(P_i, X; \mathbf{sSet}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{s}\mathcal{C}}(P, X; \mathbf{sSet}) \arrow[d] \\
\mathrm{Hom}_{\mathrm{s}\mathcal{C}}(P, Y; \mathbf{sSet}) \arrow[r] & \prod \mathrm{Hom}_{\mathrm{s}\mathcal{C}}(P, Y; \mathbf{sSet}) \arrow[r] & \mathrm{Hom}_{\mathrm{s}\mathcal{C}}(P, Y; \mathbf{sSet})
\end{tikzcd}$$

Since weak equivalences and fibrations of simplicial sets are stable under products, we obtain that the map $\mathrm{Hom}_{\mathrm{s}\mathcal{C}}(P, f; \mathbf{sSet})$ is a fibration (resp. a weak equivalence). $\square$

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